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Coincidence Lemma and Substitution Lemma[1]

Patrick Braselmann
University of Bonn

Peter Koepke
University of Bonn

Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349–360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag New York Inc. The present article establishes further concepts of substitution of a variable for a variable in a first-order formula. The main result is the substitution lemma. The contents of this article correspond to Chapter III par. 5, 5.1 Coincidence Lemma and Chapter III par. 8, 8.3 Substitution Lemma of Ebbinghaus, Flum, Thomas.

MML Identifier: `SUBLEMMA`.

The articles [13], [7], [15], [1], [4], [9], [8], [10], [3], [18], [6], [16], [19], [5], [12], [17], [11], [14], and [2] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: a, b are sets, i, k are natural numbers, p, q are elements of CQC-WFF, x, y are bound variables, A is a non empty set, J is an interpretation of A, v, w are elements of $\boldsymbol{V}(A)$, P, P' are

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k-ary predicate symbols, l_1, l'_1 are variables lists of k, l_2 is a finite sequence of elements of Var, S_1, S'_1 are CQC-substitutions, and S, S_2, S_3 are elements of CQC-Sub-WFF.

Next we state two propositions:

(1) For all functions f, g, h, h_1, h_2 such that $\mathrm{dom}\, h_1 \subseteq \mathrm{dom}\, h$ and $\mathrm{dom}\, h_2 \subseteq \mathrm{dom}\, h$ holds $f {+\cdot} g {+\cdot} h = f {+\cdot} h_1 {+\cdot} (g {+\cdot} h_2) {+\cdot} h$.

(2) For every function v_1 such that $x \in \mathrm{dom}\, v_1$ holds $v_1 \restriction (\mathrm{dom}\, v_1 \setminus \{x\}) {+\cdot} (x \longmapsto v_1(x)) = v_1$.

Let us consider A. A value substitution of A is a partial function from BoundVar to A.

In the sequel v_2, v_1, v_3 are value substitutions of A.

Let us consider A, v, v_2. The functor $v(v_2)$ yields an element of $\boldsymbol{V}(A)$ and is defined by:

(Def. 1) $v(v_2) = v {+\cdot} v_2$.

Let us consider S. Then $S_{\mathbf{1}}$ is an element of CQC-WFF.

Let us consider S, A, v. The functor $\mathrm{ValS}(v, S)$ yielding a value substitution of A is defined by:

(Def. 2) $\mathrm{ValS}(v, S) = ({}^{@}(S_{\mathbf{2}})) \cdot v$.

The following proposition is true

(3) If S is sub-verum, then $\mathrm{CQCSub}(S) = \mathrm{VERUM}$.

Let us consider S, A, v, J. The predicate $J, v \models S$ is defined as follows:

(Def. 3) $J, v \models S_{\mathbf{1}}$.

The following propositions are true:

(4) If S is sub-verum, then for every v holds $J, v \models \mathrm{CQCSub}(S)$ iff $J, v(\mathrm{ValS}(v, S)) \models S$.

(5) If $i \in \mathrm{dom}\, l_1$, then $l_1(i)$ is a bound variable.

(6) If S is sub-atomic, then $\mathrm{CQCSub}(S) = \mathrm{PredSym}(S_{\mathbf{1}})[\text{CQC-Subst}(\mathrm{SubArguments}(S), S_{\mathbf{2}})]$.

(7) If $\mathrm{SubArguments}(\mathrm{SubP}(P, l_1, S_1)) = \mathrm{SubArguments}(\mathrm{SubP}(P', l'_1, S'_1))$, then $l_1 = l'_1$.

(8) $\mathrm{SubArguments}(\mathrm{SubP}(P, l_1, S_1)) = l_1$.

Let us consider k, P, l_1, S_1. Then $\mathrm{SubP}(P, l_1, S_1)$ is an element of CQC-Sub-WFF.

We now state three propositions:

(9) $\mathrm{CQCSub}(\mathrm{SubP}(P, l_1, S_1)) = P[\text{CQC-Subst}(l_1, S_1)]$.

(10) $P[\text{CQC-Subst}(l_1, S_1)]$ is an element of CQC-WFF.

(11) $\text{CQC-Subst}(l_1, S_1)$ is a variables list of k.

Let us consider k, l_1, S_1. Then $\text{CQC-Subst}(l_1, S_1)$ is a variables list of k.

One can prove the following propositions:

(12) If $x \notin \mathrm{dom}(S_{\mathbf{2}})$, then $v(\mathrm{ValS}(v, S))(x) = v(x)$.

(13) If $x \in \mathrm{dom}(S_{\mathbf{2}})$, then $v(\mathrm{ValS}(v, S))(x) = (\mathrm{ValS}(v, S))(x)$.

(14) $v(\mathrm{ValS}(v, \mathrm{SubP}(P, l_1, S_1))) * l_1 = v * \text{CQC-Subst}(l_1, S_1)$.

(15) $(\mathrm{SubP}(P, l_1, S_1))_{\mathbf{1}} = P[l_1]$.

(16) For every v holds $J, v \models \mathrm{CQCSub}(\mathrm{SubP}(P, l_1, S_1))$ iff $J, v(\mathrm{ValS}(v, \mathrm{SubP}(P, l_1, S_1))) \models \mathrm{SubP}(P, l_1, S_1)$.

(17) $(\mathrm{SubNot}(S))_{\mathbf{1}} = \neg(S_{\mathbf{1}})$ and $(\mathrm{SubNot}(S))_{\mathbf{2}} = S_{\mathbf{2}}$.

Let us consider S. Then $\mathrm{SubNot}(S)$ is an element of CQC-Sub-WFF.

We now state three propositions:

(18) $J, v(\mathrm{ValS}(v, S)) \not\models S$ iff $J, v(\mathrm{ValS}(v, S)) \models \mathrm{SubNot}(S)$.

(19) $\mathrm{ValS}(v, S) = \mathrm{ValS}(v, \mathrm{SubNot}(S))$.

(20) If for every v holds $J, v \models \mathrm{CQCSub}(S)$ iff $J, v(\mathrm{ValS}(v, S)) \models S$, then for every v holds $J, v \models \mathrm{CQCSub}(\mathrm{SubNot}(S))$ iff $J, v(\mathrm{ValS}(v, \mathrm{SubNot}(S))) \models \mathrm{SubNot}(S)$.

Let us consider S_2, S_3. Let us assume that $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$. The functor $\mathrm{CQCSubAnd}(S_2, S_3)$ yielding an element of CQC-Sub-WFF is defined as follows:

(Def. 4) $\mathrm{CQCSubAnd}(S_2, S_3) = \mathrm{SubAnd}(S_2, S_3)$.

Next we state several propositions:

(21) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $(\mathrm{CQCSubAnd}(S_2, S_3))_{\mathbf{1}} = (S_2)_{\mathbf{1}} \wedge (S_3)_{\mathbf{1}}$ and $(\mathrm{CQCSubAnd}(S_2, S_3))_{\mathbf{2}} = (S_2)_{\mathbf{2}}$.

(22) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $(\mathrm{CQCSubAnd}(S_2, S_3))_{\mathbf{2}} = (S_2)_{\mathbf{2}}$.

(23) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{ValS}(v, S_2) = \mathrm{ValS}(v, \mathrm{CQCSubAnd}(S_2, S_3))$ and $\mathrm{ValS}(v, S_3) = \mathrm{ValS}(v, \mathrm{CQCSubAnd}(S_2, S_3))$.

(24) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $\mathrm{CQCSub}(\mathrm{CQCSubAnd}(S_2, S_3)) = \mathrm{CQCSub}(S_2) \wedge \mathrm{CQCSub}(S_3)$.

(25) If $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$, then $J, v(\mathrm{ValS}(v, S_2)) \models S_2$ and $J, v(\mathrm{ValS}(v, S_3)) \models S_3$ iff $J, v(\mathrm{ValS}(v, \mathrm{CQCSubAnd}(S_2, S_3))) \models \mathrm{CQCSubAnd}(S_2, S_3)$.

(26) Suppose $(S_2)_{\mathbf{2}} = (S_3)_{\mathbf{2}}$ and for every v holds $J, v \models \mathrm{CQCSub}(S_2)$ iff $J, v(\mathrm{ValS}(v, S_2)) \models S_2$ and for every v holds $J, v \models \mathrm{CQCSub}(S_3)$ iff $J, v(\mathrm{ValS}(v, S_3)) \models S_3$. Let given v. Then $J, v \models \mathrm{CQCSub}(\mathrm{CQCSubAnd}(S_2, S_3))$ if and only if $J, v(\mathrm{ValS}(v, \mathrm{CQCSubAnd}(S_2, S_3))) \models \mathrm{CQCSubAnd}(S_2, S_3)$.

In the sequel B is an element of [: QC-Sub-WFF, BoundVar :] and S_4 is a second q.-component of B.

The following proposition is true

(27) If B is quantifiable, then $(\mathrm{SubAll}(B, S_4))_{\mathbf{1}} = \forall_{B_{\mathbf{2}}}((B_{\mathbf{1}})_{\mathbf{1}})$ and $(\mathrm{SubAll}(B, S_4))_{\mathbf{2}} = S_4$.

Let B be an element of [: QC-Sub-WFF, BoundVar :]. We say that B is CQC-WFF-like if and only if:

(Def. 5) $B_{\mathbf{1}} \in$ CQC-Sub-WFF.

Let us observe that there exists an element of [: QC-Sub-WFF, BoundVar :] which is CQC-WFF-like.

Let us consider S, x. Then $\langle S,\, x\rangle$ is a CQC-WFF-like element of [: QC-Sub-WFF, BoundVar :].

In the sequel B denotes a CQC-WFF-like element of [: QC-Sub-WFF, BoundVar :], x_1 denotes a second q.-component of $\langle S,\, x\rangle$, and S_4 denotes a second q.-component of B.

Let us consider B. Then $B_{\mathbf{1}}$ is an element of CQC-Sub-WFF.

Let us consider B, S_4. Let us assume that B is quantifiable. The functor CQCSubAll(B, S_4) yields an element of CQC-Sub-WFF and is defined as follows:

(Def. 6) CQCSubAll(B, S_4) = SubAll(B, S_4).

We now state the proposition

(28) If B is quantifiable, then CQCSubAll(B, S_4) is sub-universal.

Let us consider S. Let us assume that S is sub-universal. The functor CQCSubScope(S) yielding an element of CQC-Sub-WFF is defined as follows:

(Def. 7) CQCSubScope(S) = SubScope(S).

Let us consider S_2, p. Let us assume that S_2 is sub-universal and p = CQCSub(CQCSubScope(S_2)). The functor CQCQuant(S_2, p) yielding an element of CQC-WFF is defined as follows:

(Def. 8) CQCQuant(S_2, p) = Quant(S_2, p).

The following two propositions are true:

(29) If S is sub-universal, then CQCSub(S) = CQCQuant(S, CQCSub(CQCSubScope(S))).

(30) If B is quantifiable, then CQCSubScope(CQCSubAll(B, S_4)) = $B_{\mathbf{1}}$.

2. The Substitution Lemma

The following propositions are true:

(31) If $\langle S,\, x\rangle$ is quantifiable, then CQCSubScope(CQCSubAll($\langle S,\, x\rangle, x_1$)) = S and CQCQuant(CQCSubAll($\langle S,\, x\rangle, x_1$), CQCSub(CQCSubScope (CQCSubAll($\langle S,\, x\rangle, x_1$)))) = CQCQuant(CQCSubAll($\langle S,\, x\rangle, x_1$), CQCSub$(S)$).

(32) If $\langle S,\, x\rangle$ is quantifiable, then CQCQuant(CQCSubAll($\langle S,\, x\rangle, x_1$), CQCSub$(S)$) = $\forall_{\text{S-Bound}(^{@}\text{CQCSubAll}(\langle S, x\rangle, x_1))}$ CQCSub(S).

(33) If $x \in \mathrm{dom}(S_{\mathbf{2}})$, then $v((^{@}(S_{\mathbf{2}}))(x)) = v(\mathrm{ValS}(v, S))(x)$.

(34) If $x \in \mathrm{dom}(^{@}(S_{\mathbf{2}}))$, then $(^{@}(S_{\mathbf{2}}))(x)$ is a bound variable.

(35) [: WFF, vSUB :] $\subseteq$ dom QSub.

In the sequel B_1 denotes an element of $[\!:$ QC-Sub-WFF, BoundVar $:\!]$ and S_5 denotes a second q.-component of B_1.

We now state a number of propositions:

(36) If B is quantifiable and B_1 is quantifiable and $\mathrm{SubAll}(B, S_4) = \mathrm{SubAll}(B_1, S_5)$, then $B_{\mathbf{2}} = (B_1)_{\mathbf{2}}$ and $S_4 = S_5$.

(37) If B is quantifiable and B_1 is quantifiable and $\mathrm{CQCSubAll}(B, S_4) = \mathrm{SubAll}(B_1, S_5)$, then $B_{\mathbf{2}} = (B_1)_{\mathbf{2}}$ and $S_4 = S_5$.

(38) If $\langle S, x\rangle$ is quantifiable, then $\mathrm{SubBound}(\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) = x$.

(39) If $\langle S, x\rangle$ is quantifiable and $x \in \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$, then $\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$ and $\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) \notin \mathrm{BoundVars}(S_{\mathbf{1}})$.

(40) If $\langle S, x\rangle$ is quantifiable and $x \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$, then $\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$.

(41) If $\langle S, x\rangle$ is quantifiable, then $\text{S-Bound}({}^{@}\mathrm{CQCSubAll}(\langle S, x\rangle, x_1)) \notin \mathrm{rng}\,\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$.

(42) If $\langle S, x\rangle$ is quantifiable, then $S_{\mathbf{2}} = \mathrm{ExpandSub}(x, S_{\mathbf{1}}, \mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1))$.

(43) $\mathrm{snb}(\mathrm{VERUM}) \subseteq \mathrm{BoundVars}(\mathrm{VERUM})$.

(44) $\mathrm{snb}(P[l_1]) \subseteq \mathrm{BoundVars}(P[l_1])$.

(45) If $\mathrm{snb}(p) \subseteq \mathrm{BoundVars}(p)$, then $\mathrm{snb}(\neg p) \subseteq \mathrm{BoundVars}(\neg p)$.

(46) If $\mathrm{snb}(p) \subseteq \mathrm{BoundVars}(p)$ and $\mathrm{snb}(q) \subseteq \mathrm{BoundVars}(q)$, then $\mathrm{snb}(p \wedge q) \subseteq \mathrm{BoundVars}(p \wedge q)$.

(47) If $\mathrm{snb}(p) \subseteq \mathrm{BoundVars}(p)$, then $\mathrm{snb}(\forall_x p) \subseteq \mathrm{BoundVars}(\forall_x p)$.

(48) For every p holds $\mathrm{snb}(p) \subseteq \mathrm{BoundVars}(p)$.

Let us consider A, let a be an element of A, and let us consider x. The functor $x \restriction a$ yields a value substitution of A and is defined as follows:

(Def. 9) $x \restriction a = x \longmapsto a$.

In the sequel a denotes an element of A.

The following propositions are true:

(49) If $x \neq b$, then $v(x \restriction a)(b) = v(b)$.

(50) If $x = y$, then $v(x \restriction a)(y) = a$.

(51) $J, v \models \forall_x p$ iff for every a holds $J, v(x \restriction a) \models p$.

Let us consider S, x, x_1, A, v. The functor $\mathrm{NExVal}(v, S, x, x_1)$ yielding a value substitution of A is defined as follows:

(Def. 10) $\mathrm{NExVal}(v, S, x, x_1) = ({}^{@}\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)) \cdot v$.

Let us consider A and let v, w be value substitutions of A. Then $v{+}{\cdot}w$ is a value substitution of A.

One can prove the following propositions:

(52) If $\langle S, x\rangle$ is quantifiable and $x \in \operatorname{rng} \mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$, then S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$)) $= \mathrm{x}_{\mathrm{upVar}(\mathrm{RestrictSub}(x,\forall_x(S_{\mathbf{1}}),x_1),S_{\mathbf{1}})}$.

(53) If $\langle S, x\rangle$ is quantifiable and $x \notin \operatorname{rng} \mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$, then S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$)) $= x$.

(54) If $\langle S, x\rangle$ is quantifiable, then for every a holds ValS(v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S$) = NExVal(v(S-Bound ($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S, x, x_1)+\cdot x\restriction a$ and dom RestrictSub($x, \forall_x(S_{\mathbf{1}}), x_1$) misses $\{x\}$.

(55) Suppose $\langle S, x\rangle$ is quantifiable. Then for every a holds J, v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a$)(ValS(v(S-Bound($^{@}$CQCSubAll ($\langle S, x\rangle, x_1$))$\restriction a), S$)) $\models S$ if and only if for every a holds J, v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a$)(NExVal(v(S-Bound ($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S, x, x_1)+\cdot x\restriction a) \models S$.

(56) If $\langle S, x\rangle$ is quantifiable, then for every a holds NExVal(v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S, x, x_1$) = NExVal($v, S, x, x_1$).

(57) Suppose $\langle S, x\rangle$ is quantifiable. Then for every a holds J, v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a$)(NExVal(v(S-Bound ($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a), S, x, x_1)+\cdot x\restriction a) \models S$ if and only if for every a holds J, v(S-Bound($^{@}$CQCSubAll($\langle S, x\rangle, x_1$))$\restriction a$)(NExVal(v, S, x, x_1) $+\cdot x\restriction a) \models S$.

3. The Coincidence Lemma

The following propositions are true:

(58) If $\operatorname{rng} l_2 \subseteq$ BoundVar, then $\mathrm{snb}(l_2) = \operatorname{rng} l_2$.

(59) $\operatorname{dom} v =$ BoundVar and $\operatorname{dom}(x\restriction a) = \{x\}$.

(60) $v * l_1 = l_1 \cdot (v\restriction \mathrm{snb}(l_1))$.

(61) For all v, w such that $v\restriction \mathrm{snb}(P[l_1]) = w\restriction \mathrm{snb}(P[l_1])$ holds $J, v \models P[l_1]$ iff $J, w \models P[l_1]$.

(62) Suppose that for all v, w such that $v\restriction \mathrm{snb}(p) = w\restriction \mathrm{snb}(p)$ holds $J, v \models p$ iff $J, w \models p$. Let given v, w. If $v\restriction \mathrm{snb}(\neg p) = w\restriction \mathrm{snb}(\neg p)$, then $J, v \models \neg p$ iff $J, w \models \neg p$.

(63) Suppose that
  (i) for all v, w such that $v\restriction \mathrm{snb}(p) = w\restriction \mathrm{snb}(p)$ holds $J, v \models p$ iff $J, w \models p$, and
  (ii) for all v, w such that $v\restriction \mathrm{snb}(q) = w\restriction \mathrm{snb}(q)$ holds $J, v \models q$ iff $J, w \models q$.

Let given v, w. If $v\restriction \mathrm{snb}(p \wedge q) = w\restriction \mathrm{snb}(p \wedge q)$, then $J, v \models p \wedge q$ iff $J, w \models p \wedge q$.

(64) For every set X such that $X \subseteq$ BoundVar holds $\operatorname{dom}(v\restriction X) = \operatorname{dom}(v(x\restriction a)\restriction X)$ and $\operatorname{dom}(v\restriction X) = X$.

(65) If $v\restriction \operatorname{snb}(p) = w\restriction \operatorname{snb}(p)$, then $v(x\restriction a)\restriction \operatorname{snb}(p) = w(x\restriction a)\restriction \operatorname{snb}(p)$.

(66) $\operatorname{snb}(p) \subseteq \operatorname{snb}(\forall_x p) \cup \{x\}$.

(67) If $v\restriction(\operatorname{snb}(p) \setminus \{x\}) = w\restriction(\operatorname{snb}(p) \setminus \{x\})$, then $v(x\restriction a)\restriction \operatorname{snb}(p) = w(x\restriction a)\restriction \operatorname{snb}(p)$.

(68) Suppose that for all v, w such that $v\restriction \operatorname{snb}(p) = w\restriction \operatorname{snb}(p)$ holds $J, v \models p$ iff $J, w \models p$. Let given v, w. If $v\restriction \operatorname{snb}(\forall_x p) = w\restriction \operatorname{snb}(\forall_x p)$, then $J, v \models \forall_x p$ iff $J, w \models \forall_x p$.

(69) For all v, w such that $v\restriction \operatorname{snb}(\mathrm{VERUM}) = w\restriction \operatorname{snb}(\mathrm{VERUM})$ holds $J, v \models \mathrm{VERUM}$ iff $J, w \models \mathrm{VERUM}$.

(70) For every p and for all v, w such that $v\restriction \operatorname{snb}(p) = w\restriction \operatorname{snb}(p)$ holds $J, v \models p$ iff $J, w \models p$.

(71) If $\langle S, x\rangle$ is quantifiable, then $v(\text{S-Bound}({}^{@}\text{CQCSubAll}(\langle S, x\rangle, x_1))\restriction a)(\text{NExVal}(v, S, x, x_1){+}\cdot x\restriction a)\restriction \operatorname{snb}(S_{\mathbf{1}}) = v(\text{NExVal}(v, S, x, x_1){+}\cdot x\restriction a)\restriction \operatorname{snb}(S_{\mathbf{1}})$.

(72) If $\langle S, x\rangle$ is quantifiable, then for every a holds $J, v(\text{S-Bound}({}^{@}\text{CQCSubAll}(\langle S, x\rangle, x_1))\restriction a)(\text{NExVal}(v, S, x, x_1){+}\cdot x\restriction a) \models S$ iff for every a holds $J, v(\text{NExVal}(v, S, x, x_1){+}\cdot x\restriction a) \models S$.

(73) $\operatorname{dom} \text{NExVal}(v, S, x, x_1) = \operatorname{dom} \text{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)$.

(74) If $\langle S, x\rangle$ is quantifiable, then $v(\text{NExVal}(v, S, x, x_1){+}\cdot x\restriction a) = v(\text{NExVal}(v, S, x, x_1))(x\restriction a)$.

(75) If $\langle S, x\rangle$ is quantifiable, then for every a holds $J, v(\text{NExVal}(v, S, x, x_1){+}\cdot x\restriction a) \models S$ iff for every a holds $J, v(\text{NExVal}(v, S, x, x_1))(x\restriction a) \models S$.

(76) For every a holds $J, v(\text{NExVal}(v, S, x, x_1))(x\restriction a) \models S$ iff for every a holds $J, v(\text{NExVal}(v, S, x, x_1))(x\restriction a) \models S_{\mathbf{1}}$.

(77) Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(\mathrm{VERUM})$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$. Then $J, v(v_2) \models \mathrm{VERUM}$ if and only if $J, v(v_2{+}\cdot v_1{+}\cdot v_3) \models \mathrm{VERUM}$.

(78) Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(l_1)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$. Then $v(v_2) * l_1 = v(v_2{+}\cdot v_1{+}\cdot v_3) * l_1$.

(79) Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(P[l_1])$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$. Then $J, v(v_2) \models P[l_1]$ if and only if $J, v(v_2{+}\cdot v_1{+}\cdot v_3) \models P[l_1]$.

(80) Suppose that for all v, v_2, v_1, v_3 such that for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(p)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) =$

$v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$ holds $J, v(v_2) \models p$ iff $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p$. Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(\neg p)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$. Then $J, v(v_2) \models \neg p$ if and only if $J, v(v_2 +\cdot v_1 +\cdot v_3) \models \neg p$.

(81) Suppose that

(i) for all v, v_2, v_1, v_3 such that for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(p)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$ holds $J, v(v_2) \models p$ iff $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p$, and

(ii) for all v, v_2, v_1, v_3 such that for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(q)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$ holds $J, v(v_2) \models q$ iff $J, v(v_2 +\cdot v_1 +\cdot v_3) \models q$.

Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(p \wedge q)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$. Then $J, v(v_2) \models p \wedge q$ if and only if $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p \wedge q$.

(82) If for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(\forall_x p)$, then for every y such that $y \in \operatorname{dom} v_1 \setminus \{x\}$ holds $y \notin \operatorname{snb}(p)$.

(83) Let v_1 be a function. Suppose for every y such that $y \in \operatorname{dom} v_1$ holds $v_1(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_1$. Let given y. If $y \in \operatorname{dom} v_1 \setminus \{x\}$, then $(v_1 \restriction (\operatorname{dom} v_1 \setminus \{x\}))(y) = v(v_2)(y)$.

(84) Suppose that for all v, v_2, v_1, v_3 such that for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(p)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$ holds $J, v(v_2) \models p$ iff $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p$. Let given v, v_2, v_1, v_3. Suppose for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(\forall_x p)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$. Then $J, v(v_2) \models \forall_x p$ if and only if $J, v(v_2 +\cdot v_1 +\cdot v_3) \models \forall_x p$.

(85) Let given p and given v, v_2, v_1, v_3. Suppose for every y such that $y \in \operatorname{dom} v_1$ holds $y \notin \operatorname{snb}(p)$ and for every y such that $y \in \operatorname{dom} v_3$ holds $v_3(y) = v(y)$ and $\operatorname{dom} v_2$ misses $\operatorname{dom} v_3$. Then $J, v(v_2) \models p$ if and only if $J, v(v_2 +\cdot v_1 +\cdot v_3) \models p$.

Let us consider p. The functor $\operatorname{RSub1} p$ yields a set and is defined by:

(Def. 11) $b \in \operatorname{RSub1} p$ iff there exists x such that $x = b$ and $x \notin \operatorname{snb}(p)$.

Let us consider p, S_1. The functor $\operatorname{RSub2}(p, S_1)$ yielding a set is defined as follows:

(Def. 12) $b \in \operatorname{RSub2}(p, S_1)$ iff there exists x such that $x = b$ and $x \in \operatorname{snb}(p)$ and $x = ({}^{@}S_1)(x)$.

Next we state several propositions:

(86) $\operatorname{dom}(({}^{@}S_1) \restriction \operatorname{RSub1} p)$ misses $\operatorname{dom}(({}^{@}S_1) \restriction \operatorname{RSub2}(p, S_1))$.

(87) ${}^{@}\mathrm{RestrictSub}(x, \forall_x p, S_1) =$
$({}^{@}S_1) \setminus (({}^{@}S_1) \restriction \mathrm{RSub1}\, \forall_x p + \cdot ({}^{@}S_1) \restriction \mathrm{RSub2}(\forall_x p, S_1))$.

(88) $\mathrm{dom}({}^{@}\mathrm{RestrictSub}(x, p, S_1))$ misses
$\mathrm{dom}(({}^{@}S_1) \restriction \mathrm{RSub1}\, p) \cup \mathrm{dom}(({}^{@}S_1) \restriction \mathrm{RSub2}(p, S_1))$.

(89) If $\langle S, x \rangle$ is quantifiable, then ${}^{@}((\mathrm{CQCSubAll}(\langle S, x \rangle, x_1))_{\mathbf{2}}) =$ $({}^{@}\mathrm{RestrictSub}(x, \forall_x(S_{\mathbf{1}}), x_1)) + \cdot ({}^{@}x_1) \restriction \mathrm{RSub1}\, \forall_x(S_{\mathbf{1}}) + \cdot ({}^{@}x_1) \restriction \mathrm{RSub2}$ $(\forall_x(S_{\mathbf{1}}), x_1)$.

(90) Suppose $\langle S, x \rangle$ is quantifiable. Then there exist v_1, v_3 such that
- (i) for every y such that $y \in \mathrm{dom}\, v_1$ holds $y \notin \mathrm{snb}(\forall_x(S_{\mathbf{1}}))$,
- (ii) for every y such that $y \in \mathrm{dom}\, v_3$ holds $v_3(y) = v(y)$,
- (iii) $\mathrm{dom}\, \mathrm{NExVal}(v, S, x, x_1)$ misses $\mathrm{dom}\, v_3$, and
- (iv) $v(\mathrm{ValS}(v, \mathrm{CQCSubAll}(\langle S, x \rangle, x_1))) = v(\mathrm{NExVal}(v, S, x, x_1) + \cdot v_1 + \cdot v_3)$.

(91) If $\langle S, x \rangle$ is quantifiable, then for every v holds $J, v(\mathrm{NExVal}(v, S, x, x_1)) \models \forall_x(S_{\mathbf{1}})$ iff $J, v(\mathrm{ValS}(v, \mathrm{CQCSubAll}(\langle S, x \rangle, x_1))) \models \mathrm{CQCSubAll}(\langle S, x \rangle, x_1)$.

(92) Suppose $\langle S, x \rangle$ is quantifiable and for every v holds $J, v \models \mathrm{CQCSub}(S)$ iff $J, v(\mathrm{ValS}(v, S)) \models S$. Let given v. Then $J, v \models \mathrm{CQCSub}(\mathrm{CQCSubAll}(\langle S, x \rangle, x_1))$ if and only if $J, v(\mathrm{ValS}(v, \mathrm{CQCSubAll}(\langle S, x \rangle, x_1))) \models \mathrm{CQCSubAll}(\langle S, x \rangle, x_1)$.

The scheme *SubCQCInd1* concerns a unary predicate $\mathcal{P}$, and states that:

For every S holds $\mathcal{P}[S]$

provided the following condition is met:

- Let S, S' be elements of CQC-Sub-WFF, x be a bound variable, S_4 be a second q.-component of $\langle S, x \rangle$, k be a natural number, l_1 be a variables list of k, P be a k-ary predicate symbol, and e be an element of vSUB. Then
  - (i) $\mathcal{P}[\mathrm{SubP}(P, l_1, e)]$,
  - (ii) if S is sub-verum, then $\mathcal{P}[S]$,
  - (iii) if $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{SubNot}(S)]$,
  - (iv) if $S_{\mathbf{2}} = S'_{\mathbf{2}}$ and $\mathcal{P}[S]$ and $\mathcal{P}[S']$, then $\mathcal{P}[\mathrm{CQCSubAnd}(S, S')]$, and
  - (v) if $\langle S, x \rangle$ is quantifiable and $\mathcal{P}[S]$, then $\mathcal{P}[\mathrm{CQCSubAll}(\langle S, x \rangle, S_4)]$.

Next we state the proposition

(93) For all S, v holds $J, v \models \mathrm{CQCSub}(S)$ iff $J, v(\mathrm{ValS}(v, S)) \models S$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Patrick Braselmann and Peter Koepke. Substitution in first-order formulas: Elementary properties. *Formalized Mathematics*, 13(**1**):5–15, 2005.

[3] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Czesław Byliński and Grzegorz Bancerek. Variables in formulae of the first order language. *Formalized Mathematics*, 1(**3**):459–469, 1990.
[9] Piotr Rudnicki and Andrzej Trybulec. A first order language. *Formalized Mathematics*, 1(**2**):303–311, 1990.
[10] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[11] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[12] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[16] Edmund Woronowicz. Interpretation and satisfiability in the first order logic. *Formalized Mathematics*, 1(**4**):739–743, 1990.
[17] Edmund Woronowicz. Many–argument relations. *Formalized Mathematics*, 1(**4**):733–737, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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