Embedding Normal Forms and Π^1_1–Determinacy

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Abstract

We give a new proof of D. Martin's theorem that $\mathbf{\Pi}^1_1$–sets of reals are determined if there is a measurable cardinal. The argument is based on representing $\mathbf{\Pi}^1_1$–sets using systems of elementary embeddings of models of set theory.

1 Games on Trees

We consider games whose positions are finite sequences and where two players called I and II alternately try to lengthen a position by one move. Thereby, they determine a maximal path through the tree of positions. Player I's aim is to get this path into a fixed winning set while player II tries to prevent this.
Accordingly we define: A *tree* is a nonempty set of finite sequences, $T \subseteq {}^{<\omega}V$, closed under the formation of initial segments. T is partially ordered by $\subseteq$. A *path* through T is a sequence p of length $\leq \omega$ such that $\forall n < \omega \; : p \uparrow n \in T$; p is *maximal* if there is no path through T properly extending p. A maximal path through T is also called a *play* on T. A play $p = (a_0, a_1, a_2, a_3, \ldots)$ is sometimes represented in the form

I	a_0		a_2		$\ldots$	
II		a_1		a_3		$\ldots$

to indicate that I makes the move a_0, then II answers a_1, I makes the move a_2, etc. Let $[T]$ denote the set of plays of T.

A *game* $G(T,A)$ on T is given by a set $A \subseteq [T]$ of *winning plays* for I$\dot{\text{W}}$e say that I *wins* the play p in the game $G(T,A)$, if $p \in A$; II *wins* if $p \in [T] \setminus A$. The obvious question then is whether one of the players possesses a winning strategy in this game. A *strategy* on T is a function $\sigma : T \to T$ such that

$$\forall t \in T \ (t \text{ is not maximal in } (T, \subseteq) \Rightarrow (\sigma(t) \supseteq t \wedge |\sigma(t)| = |t| + 1)),$$

where $|t|$ denotes the length of the finite sequence t. A strategy $\sigma : T \to T$ is a *winning strategy* for I in the game $G(T,A)$, if

$$\forall p \in [T] \ [(\forall 2n+1 \leq |p| (p \upharpoonright 2n+1 = \sigma(p \upharpoonright 2n))) \Rightarrow p \in A].$$

Similarly, σ is a *winning strategy* for II if

$$\forall p \in [T][(\forall 2n+2 \leq |p| (p \upharpoonright 2n+2 = \sigma(p \upharpoonright 2n+1))) \Rightarrow p \in [T] \setminus A].$$

I and II cannot both have winning strategies in $G(T,A)$. $G(T,A)$ is *determined*, if one of the players has a winning strategy in $G(T,A)$.
We are mainly interested in games on the real numbers. Here, T is the tree ${}^{<\omega}\omega$ of finite sequences of natural numbers. We identify $[T] = {}^{\omega}\omega$ with the set $\mathbf{R}$ of real numbers. A set $A \subseteq \mathbf{R}$ is called *determined* if G(A)= $G({}^{<\omega}\omega, A)$ is determined. $\mathbf{\Pi}^1_1$*–determinacy* is the statement that every $\mathbf{\Pi}^1_1$–set of reals is determined. The determinacy of pointclasses like $\mathbf{\Pi}^1_1$ has profound implications for the descriptive set theory of projective sets (see Moschovakis [5]). In the course of this article we shall prove the following theorem of D. Martin [3]:
(1.1) Theorem. *If there is a measurable cardinal then* $\mathbf{\Pi}^1_1$*–determinacy holds.*
The original proof employed indiscernibles derived from the partition properties of measurable cardinals. The argument given here emphasizes the view that measurable cardinals are best characterized by elementary embeddings of the universe, as are many large cardinals above measurables. Correspondingly, we shall represent $\mathbf{\Pi}^1_1$–sets by *embedding normal forms* constructed from iterated ultrapowers of the universe and natural embeddings among them.

2 An Auxiliary Game

Consider the game G(A)= $G({}^{<\omega}\omega, A)$, where $A \subseteq \mathbf{R}$. An auxiliary game $G^*(A)$ will be defined by adding some "side moves" for I and a system of rules such that if I satisfies all the rules then I has also produced a winning play for the original game $G(A)$. The crucial point will be that $G^*(A)$, due to its simple (topological) nature, will be determined. Let T^* consist of all finite sequences of the form

$$((a_0, f_0), a_1, (a_2, f_2), a_3, \ldots, (a_{2n}, f_{2n}))$$

or

$$((a_0, f_0), a_1, (a_2, f_2), a_3, \ldots, (a_{2n}, f_{2n}), a_{2n+1})$$

such that (2.1) – (2.3) hold:

(2.1) $a_j \in \omega$, for $j < 2n+2$;

(2.2) $f_{2j} : \mathbf{R} \to \theta$, for $j \leq n$;

for some fixed sufficiently large ordinal θ; the size of θ will only become important in Chapter 4, and we shall give an adequate lower bound for θ in (4.2);

(2.3) $\forall x \in \mathbf{R} \setminus A (x \supseteq (a_0, \ldots, a_{2i+2}) \Rightarrow f_{2i}(x) > f_{2i+2}(x))$, for $i < n$.

A play on T^* may be represented as

I	$a_0,\ f_0$		$a_2,\ f_2$		$\ldots$		
II		a_1		a_3		$\ldots$	

Since there is no infinite descent in the ordinals, the functions $f_0, f_2, \ldots, f_{2n}$ serve to "push away" the sequence $(a_0, a_1, \ldots)$ from $\mathbf{R} \setminus A$ and into A. Player I wins the game $G^*(A)$ if I is able to satisfy the rules (2.1) – (2.3) infinitely often. So we define formally:

$$\begin{aligned} A^* &= \{p \in [T^*] \mid p \text{ is infinite}\}, \\ G^*(A) &= G(T^*, A^*). \end{aligned}$$

(2.4) Lemma. *$G^*(A)$ is determined*

Proof. This follows from the Gale-Stewart Theorem on the determinacy of closed games [1] as A^* is closed in the natural topology on $[T^*]$. A direct argument would run as follows: Call a position a *winning position* for II if II can force a finite play starting from that position.

Now assume that II has no winning strategy in $G^*(A)$. Then the initial position $\emptyset$ is not a winning position for II. Now whenever $t \in T^*$ is of even length $2n$ and is not a winning positon for II then there must be an extension $\sigma(t) = t ^\frown (a_{2n}, f_{2n})$ such that $\sigma(t)$ is not a winning position for II.

This function σ is basically a strategy for I and if I follows σ in a play p in $G^*(A)$, then p is infinite. Hence I has a winning strategy in $G^*(A)$. $\square$

Player I is now able to turn a winning strategy for $G^*(A)$ into a winning strategy for $G(A)$ by "hiding" the side-moves $f_0, f_1, \ldots$.

(2.5) Lemma. *If σ^* is a winning strategy for I in $G^*(A)$ then I has a winning strategy in $G(A)$.*

Proof. Take σ satisfying:

$$\begin{aligned} \sigma(\emptyset) &= (a_0), &\text{if}\quad & \sigma^*(\emptyset) = ((a_0, f_0)); \\ \sigma(a_0, a_1) &= (a_0, a_1, a_2), &\text{if}\quad & \sigma^*((a_0, f_0), a_1) = ((a_0, f_0), a_1, (a_2, f_2)); \\ \sigma(a_0, a_1, a_2, a_3) &= (a_0, a_1, a_2, a_4) &\text{if}\quad & \sigma^*((a_0, f_0), a_1, (a_2, f_2), a_3) \\ & & & = ((a_0, f_0), a_1, (a_2, f_2), a_3, (a_4, f_4)); \\ & & \text{etc.} & \end{aligned}$$

Now let $p = (a_0, a_1, a_2, \ldots)$ be a play in $G(A)$ in which I follows σ. With the $f_0, f_1, \ldots$ as above, $p^* = ((a_0, f_0), a_1, (a_2, f_2), a_3, \ldots)$ is a play in $G^*(A)$ in which I follows σ^*.

If $p \in \mathbf{R} \setminus A$, then rule (2.3) implies: $f_0(p) > f_2(p) > f_4(p) > f_6(p) > \ldots$, a contradiction. Hence $p \in A$, and σ is a winning strategy for I in $G(A)$. $\square$

3 Normal Forms for $\mathbf{\Pi}^1_1$-sets

Lemmas 2.4 and 2.5 had simple proofs which did not depend on any special assumptions on A. The definability of A and the measurable cardinal only enter into the complementary argument where a winning strategy σ^* for II in $G^*(A)$ has to be transformed into a winning strategy σ for II in $G(A)$. At a position $a_0, a_1, \ldots, a_{2n}$ in $G(A)$, player II simulates playing in $G^*(A)$ by

assuming suitable functions $f_0, \ldots, f_{2n}$ for I. The strategy σ^* then yields a move a_{2n+1} for II in $G^*(A)$ which II also plays in $G(A)$. Adequate functions for the simulation of I's moves in $G^*(A)$ will be obtainable from a (sufficiently complete) *embedding normal form* for A.
So let us now assume that A is a $\mathbf{\Pi}^1_1$–set of reals. Let us also suppose that κ is a measurable cardinal in V with normal ultrafilter U. It is essentially shown in Moschovakis [5], Lemma 6G.6, that our $\mathbf{\Pi}^1_1$-set A has a representation of the following form: there is an assignment

$$s \mapsto <_s, \text{ for } s \in {}^{<\omega}\omega,$$

such that

(3.1) $<_s$ linearly orders $|s|$;

(3.2) $s \subseteq t \in {}^{<\omega}\omega \;\Rightarrow\; <_s \subseteq <_t$;

(3.3) for every $p \in \mathbf{R}$:
$p \in A$ iff $<_p := \bigcup_{n<\omega} <_{p\restriction n}$ is a wellordering of ω.

We transform this system into a commuting system of orderpreserving embeddings

$$e_{s,t} : (|s|, <) \rightarrow (|t|, <), \text{ for } s \subseteq t \in {}^{<\omega}\omega,$$

where $|s|$ and $|t|$ carry the usual order $<$. For $s \in {}^{<\omega}\omega$ let

$$h_s : (|s|, <) \leftrightarrow (|s|, <_s)$$

be $<$–$<_s$–orderpreserving. For $s \subseteq t \in {}^{<\omega}\omega$ define $e_{s,t}$ to be

$$e_{s,t} = h_t^{-1} \circ h_s.$$

The system

(3.4) $((|s|, <), e_{s,t})_{s \subseteq t \in {}^{<\omega}\omega}$

commutes, and via $(h_s)_{s\in{}^{<\omega}\omega}$ it is isomorphic to

$$((|s|,<_s),\mathrm{id}\restriction|s|)_{s\subseteq t\in{}^{<\omega}\omega}.$$

Hence, for $p\in\mathbf{R}$, the direct limit of

$$((m,<),e_{p\restriction m,p\restriction n})_{m\leq n<\omega}$$

is isomorpic to

$$<_p=\bigcup_{n\in\omega}<_{p\restriction n}\ .$$

By (3.3) we get:

(3.5) for every $p\in\mathbf{R}$:
$p\in A$ iff $\lim_{m\leq n<\omega}((m,<),e_{p\restriction m,p\restriction n})$ is wellfounded,

where the right-hand side denotes the formation of a direct limit.
From (3.5), player II could extract simulations of I's side-moves, which satisfy the rule (2.3). However, the "simulations" at various stages of the game have to be consistent in the following sence. If we employed $f_0,\ldots,f_{2n}$, to define a_{2n+1} and $f'_0,\ldots,f'_{2n},f'_{2n+2}$ to define a_{2n+3}, then the latter sequence also has to be compatible with a_{2n+1}. We need to have:

$$a_{2n+1}=\text{the last move of }\sigma^*(a_0,\ldots,a_{2n},f_0,\ldots,f_{2n})$$

and also

$$a_{2n+1}=\text{the last move of }\sigma^*(a_0,\ldots,a_{2n},f'_0,\ldots,f'_{2n})$$

(on the right-hand side, we have permuted the input sequences of the strategy to increase legibility). We obtain this compatibility by using indiscernible functions. An Ehrenfeucht-Mostowski idea is used to map the natural numbers in the system

$$((|s|,<),e_{s,t})_{s\subseteq t\in{}^{<\omega}\omega}$$

up to the canonical indiscernibles of iterated ultrapowers of V.
We recall some key facts about *iterated ultrapowers* of V, and refer the reader to Jech [2] for further details. From a normal ultrafilter U on the measurable cardinal κ one defines the following system:

$$N_0=V,\ \pi_{00}=\mathrm{id}:V\to V,\ \kappa_0=\kappa,U_0=U;$$

in the successor step set:

$N_{\alpha+1} = \text{Ult}(N_\alpha, U_\alpha)$ is the ultrapower of N_α by U_α, which is transitive;

$\pi_{\alpha,\alpha+1} : N_\alpha \longrightarrow N_{\alpha+1}$ is the natural elementary embedding into the ultrapower; $\pi_{\alpha+1,\alpha+1} = \text{id} \upharpoonright N_{\alpha+1}$; $\pi_{\gamma,\alpha+1} = \pi_{\alpha,\alpha+1} \circ \pi_{\gamma,\alpha}$, for $\gamma < \alpha$;

$\kappa_{\alpha+1} = \pi_{0,\alpha+1}(\kappa_0)$, $U_{\alpha+1} = \pi_{0,\alpha+1}(U_0)$;

for limit ordinals λ we let N_λ, $(\pi_{\alpha,\lambda})_{\alpha\leq\lambda}$ be the transitive direct limit of $(N_\alpha, \pi_{\alpha,\beta})_{\alpha\leq\beta<\lambda}$, and $\kappa_\lambda = \pi_{0,\lambda}(\kappa_0)$, $U_\lambda = \pi_{0,\lambda}(U_0)$.
The following two statements express that N_α is an Ehrenfeucht-Mostowski model for the (class sized) theory of $(V, \in)$ with constant symbols for every set $x \in V$, and the model is generated by the wellorder α:

(3.6) Each N_α is Σ_1-generated by range$(\pi_{0,\alpha}) \cup \{\kappa_i : i < \alpha\}$:

$$N_\alpha = \{\pi_{0,\alpha}(f)(\kappa_{i_1}, \ldots, \kappa_{i_n}) : n \in \omega, f : \kappa^n \to V, i_1 < \ldots < i_n < \alpha\}.$$

(3.7) The set $\{\kappa_i : i < \alpha\}$ is a set of orderindiscernibles for N_α relative to parameters from range$(\pi_{0,\alpha})$.

These facts imply the following lifting properties for orderpreserving maps:

(3.8) Let $e : \alpha \to \beta$ be (strictly) orderpreserving, $\alpha \leq \beta \in \text{On}$. Then there is a canonical elementary map

$$e^* : N_\alpha \longrightarrow N_\beta$$

defined by:

$$e^*(\pi_{0,\alpha}(f)(\kappa_{i_1}, \ldots, \kappa_{i_n})) = \pi_{0,\beta}(f)(\kappa_{e(i_1)}, \ldots, \kappa_{e(i_n)}),$$

for $n < \omega$, $f : \kappa^n \to V$, $i_1 < \ldots < i_n < \alpha$.

(3.9) If $(e_{m,n})_{m\leq n<\omega}$ is a commutative system of orderpreserving maps $e_{m,n} : m \to n$, then $(e^*_{m,n})_{m\leq n<\omega}$ commutes.

(3.10) Moreover, the system $(m, e_{m,n})_{m\leq n<\omega}$ has a wellfounded direct limit iff the system $(N_m, e^*_{m,n})_{m\leq n<\omega}$ has a wellfounded direct limit.

Proof. The system $(m, e_{m,n})_{m\le n<\omega}$ is canonically embedded into the system $(N_m, e^*_{m,n})_{m\le n<\omega}$ by the orderpreserving maps $m \to N_m$, $i \mapsto \kappa_i$. So if $(m, e_{m,n})$ has an illfounded direct limit, so has $(N_m, e^*_{m,n})$. Assume now that $(m, e_{m,n})$ has a wellfounded direct limit. Without loss of generality we can assume it to be an ordinal α; so let $(\alpha, e_m)_{m<\omega}$ be the direct limit of $(m, e_{m,n})_{m\le n<\omega}$. It is then easy to check that $(N_\alpha, e^*_m)_{m<\omega}$ is the transitive direct limit of $(N_m, e^*_{m,n})_{m\le n<\omega}$. $\square$

Let us now lift the system (3.4) by defining:

(3.11) $(M_s, \pi_{s,t})_{s\subseteq t\in {}^{<\omega}\omega}$ is given by $M_s = N_{|s|}$, $\pi_{s,t} = e^*_{s,t}$.

Facts (3.5) and (3.10) imply that

(3.12) for every $p \in \mathbf{R}$:
$p \in A$ iff $\lim_{m\le n<\omega}(M_{p\upharpoonright m}, \pi_{p\upharpoonright m, p\upharpoonright n})$ is wellfounded, hence transitive.

For the intended "simulations" we extract certain functions from the system which witness the nonwellfoundedness of paths through the system. For each $p \in \mathbf{R} \setminus A$, choose a sequence $(\gamma^p_n : n \in \omega)$ of ordinals such that

$$\pi_{p\upharpoonright m, p\upharpoonright n}(\gamma^p_m) > \gamma^p_n, \text{ for all } m < n < \omega.$$

For each $s \in {}^{<\omega}\omega$ define $w_s : \mathbf{R} \to \mathrm{On}$ by

$$w_s(p) = \begin{cases} \gamma^p_n & \text{if } s = p \upharpoonright n \text{ and } p \in \mathbf{R} \setminus A; \\ 0 & \text{else.} \end{cases}$$

Each M_s is closed under the formation of κ-sequences, since it is an iterated ultrapower of V with finite index $|s|$. Hence, $w_s \in M_s$. These functions satisfy:

(3.13) For $s \subseteq t \in {}^{<\omega}\omega$, $s \ne t$, and $p \in \mathbf{R} \setminus A$, $p \supseteq t$:

$$\pi_{s,t}(w_s)(p) > w_t(p).$$

Proof. $\pi_{s,t}(w_s)(p) = \pi_{s,t}(w_s(p)) = \pi_{s,t}(\gamma^p_{|s|}) > \gamma^p_{|t|} = w_t(p)$. $\square$

A commuting elementary system $(M_s, \pi_{s,t})_{s\subseteq t\in {}^{<\omega}\omega}$ with $M_\emptyset = V$, satisfying (3.12) is called an *embedding normal form* for A. If moreover there is a system $(w_s)_{s\in {}^{<\omega}\omega}$ of functions $w_s : \mathbf{R} \to \mathrm{On}$, with $w_s \in M_s$, satisfying (3.13), then $(w_s)_{s\in {}^{<\omega}\omega}$ is called a system of *witnesses* for the embedding normal form. We have shown above:

(3.14) Lemma. *If $A \subseteq \mathbf{R}$ is $\mathbf{\Pi}^1_1$ and if there exists a measurable cardinal then A has an embedding normal form with witnesses.*

4 Witnesses as Side-Moves

To complete the proof of Theorem (1.1), let σ^* be a winning strategy for II in $G^*(A)$, where A is the fixed $\mathbf{\Pi}^1_1$-set from above. We have to find a winning strategy for II in $G(A)$. Let

$$(M_s, \pi_{s,t}, w_s)_{s \subseteq t \in {}^{<\omega}\omega}$$

be the embedding normal form with witnesses for A as constructed in chapter 3. By (3.13), the witnesses are "descending" along the embedding normal form, and this gives arbitrary long sequences of functions satisfying the rule (2.3). We shall now describe a strategy $\sigma : {}^{<\omega}\omega \to {}^{<\omega}\omega$ which will be a winning strategy for II in $G(A)$; let $\hat{\sigma}(s)$ be the move demanded by the strategy σ on input s, i.e. $\sigma(s) = s ^\frown \hat{\sigma}(s)$; obviously we only have to define $\hat{\sigma}(s)$ for odd-length sequences s in ${}^{<\omega}\omega$ to get an appropriate σ. We set:

(4.1)

$$\begin{aligned}
\hat{\sigma}(a_0) &= \pi_{\emptyset,a_0}(\hat{\sigma}^*)(a_0, w_{a_0}); \\
\hat{\sigma}(a_0a_1a_2) &= \pi_{\emptyset,a_0a_1a_2}(\hat{\sigma}^*)(a_0, \pi_{a_0,a_0a_1a_2}(w_{a_0}), a_1, a_2, w_{a_0a_1a_2}); \\
\hat{\sigma}(s) &= \pi_{\emptyset,s}(\hat{\sigma}^*)(s, \pi_{s\upharpoonright 1,s}(w_{s\upharpoonright 1}), \pi_{s\upharpoonright 3,s}(w_{s\upharpoonright 3}), \ldots, w_s),
\end{aligned}$$

for $|s|$ odd.

Note that the sequence $(\pi_{s\upharpoonright 1,s}(w_{s\upharpoonright 1}), \pi_{s\upharpoonright 3,s}(w_{s\upharpoonright 3}), \ldots, w_s)$ is a "descending" sequence of functions which "lives" in M_s. It is therefore appropriate to apply the mapped strategy $\pi_{\emptyset,s}(\sigma^*)$. We are now able to give a lower bound for the parameter θ in (2.2). Since we want the functions $\pi_{s\upharpoonright 1,s}(w_{s\upharpoonright 1})$, $\pi_{s\upharpoonright 3,s}(w_{s\upharpoonright 3})$, …, w_s to be legitimate side-moves for I in $\pi_{\emptyset,s}(G^*(A))$, we require:

(4.2) $\theta >$ supremum of the range of $w_\emptyset$.

(4.3) Lemma. *σ is a winning strategy for II in $G(A)$.*
Proof. Let $p = (a_0, a_1, a_2, \ldots) \in \mathbf{R}$ be a play in $G(A)$ where II follows σ. Assume for a contradiction that $p \in A$. Then the direct limit

$$(M_p, \pi_{p\upharpoonright m,p})_{m\in\omega} = lim_{m \leq n < \omega}(M_{p\upharpoonright m}, \pi_{p\upharpoonright m,p\upharpoonright n})$$

is transitive by (3.12). $p = (a_0, a_1, a_2, \ldots)$ satisfies the equations (4.1). Applying the maps $\pi_{p\uparrow m,p}$ to the equations yields

$$\begin{array}{rcl} a_1 & = & \pi_{\emptyset,p}(\hat{\sigma^*})(a_0, \pi_{a_0,p}(w_{a_0})); \\ a_3 & = & \pi_{\emptyset,p}(\hat{\sigma^*})(a_0, \pi_{a_0,p}(w_{a_0}), a_1, a_2, \pi_{a_0a_1a_2,p}(w_{a_0a_1a_2})); \\ a_{2n+1} & = & \pi_{\emptyset,p}(\hat{\sigma^*})(p \uparrow 2n+1, \pi_{p\uparrow 1,p}(w_{p\uparrow 1}), \pi_{p\uparrow 3,p}(w_{p\uparrow 3}), \ldots, \pi_{p\uparrow 2n+1,p}(w_{p\uparrow 2n+1})), \end{array}$$

for $n < \omega$. The sequence of functions on the right-hand side satisfies the rule (2.3): If $x \in \mathbf{R} \setminus A$ and $p \uparrow 2n+3 \subseteq x$ then:

$$\begin{array}{rcl} \pi_{p\uparrow 2n+1,p}(w_{p\uparrow 2n+1})(x) & = & \pi_{p\uparrow 2n+3,p}(\pi_{p\uparrow 2n+1,p\uparrow 2n+3}(w_{p\uparrow 2n+1})(x)) \\ & > & \pi_{p\uparrow 2n+3,p}(w_{p\uparrow 2n+3}(x)) \text{ by (3.13),} \\ & = & \pi_{p\uparrow 2n+3,p}(w_{p\uparrow 2n+3})(x). \end{array}$$

Therefore,

$$\textbf{(4.4)}\quad \begin{array}{lccccc} \mathrm{I} & a_0, \pi_{p\uparrow 1,p}(w_{p\uparrow 1}) & & a_2, \pi_{p\uparrow 3,p}(w_{p\uparrow 3}) & & \ldots \\ \mathrm{II} & & a_1 & & a_3 & \end{array}$$

is a play in $\pi_{0,p}(G^*(A))$ in which II follows the strategy $\pi_{0,p}(\sigma^*)$ and in which the rule (2.3) is kept.
An absoluteness argument shows that a similar play must actually exist inside the model M_p. Consider, in M_p, the set $\mathcal{P}$ of all positions in $\pi_{\emptyset,p}(G^*(A))$, in which II follows the strategy $\pi_{\emptyset,p}(\sigma^*)$ (Note that a position is a finite sequence of moves). $(\mathcal{P}, \supseteq)$ is a partial order under reverse inclusion. $(\mathcal{P}, \supseteq)$ is illfounded in V as witnessed by the play (4.4). By the absoluteness of wellfoundedness between V and the *transitive* model M_p, $(\mathcal{P}, \supseteq)$ is illfounded in M_p. Hence, in M_p, there is an infinite play in $\pi_{\emptyset,p}(G^*(A))$ in which II follows the strategy $\pi_{\emptyset,p}(\sigma^*)$. Since $\pi_{\emptyset,p} : V \to M_p$ is elementary, there is, in V, an infinite play in $G^*(A)$ in which II follows the strategy σ^*. But then σ^* is not a winning strategy for II since II's aim is to make plays in $G^*(A)$ finite. Contradiction. $\square$
Let us recapitulate our result. For the game $G(A)$ we have introduced an auxiliary game $G^*(A)$ which is determined. If σ^* is a winning strategy for I in $G^*(A)$ we immediately get a winning strategy for I in $G(A)$. If σ^* is a

winning strategy for II in $G^*(A)$ the embedding normal form with witnesses which we constructed from our assumptions in chapter 3 allows us to obtain a winning strategy for II in $G(A)$. In either case, $G(A)$ is determined and we have proved Theorem (1.1).

5 Concluding remarks

(5.1) The arguments about embedding normal forms and witnesses can be modified to obtain the sharper result of D. Martin [3]: if $a^\#$ exists then every Π^1_1-set in the parameter a is determined. Instead of iterated ultrapowers one uses the model $\mathrm{L}[a]$ with its Silver-indiscernibles given by $a^\#$ as an Ehrenfeucht-Mostowski model for (3.6) and (3.7). The construction of the witnesses becomes more involved since $\mathrm{L}[a]$ is not even ω-closed.

(5.2) Our interest in embedding normal forms in determinacy proofs is motivated by the observation that the large cardinals used give embeddings of transitive models of set theory rather than combinatorial objects like measures or indiscernibles. Therefore it is tempting to work with elementary embeddings whenever possible. We have carried this out for the Martin-Steel theorem [4] (if there are infinitely many Woodin cardinals then projective determinacy holds) in an article submitted to the Journal of Symbolic Logic. That article also covers remark (5.1).

References

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