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Equivalences of Inconsistency and Henkin Models[1]

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Summary. This article is part of a series of Mizar articles which constitute a formal proof (of a basic version) of Kurt Gödel's famous completeness theorem (K. Gödel, "Die Vollständigkeit der Axiome des logischen Funktionenkalküls", Monatshefte für Mathematik und Physik 37 (1930), 349–360). The completeness theorem provides the theoretical basis for a uniform formalization of mathematics as in the Mizar project. We formalize first-order logic up to the completeness theorem as in H. D. Ebbinghaus, J. Flum, and W. Thomas, Mathematical Logic, 1984, Springer Verlag, New York Inc. The present article establishes some equivalences of inconsistency. It is proved that a countable union of consistent sets is consistent. Then the concept of a Henkin model is introduced. The contents of this article correspond to Chapter IV, par. 7 and Chapter V, par. 1 of Ebbinghaus, Flum, Thomas.

MML Identifier: `HENMODEL`.

The articles [17], [9], [19], [5], [22], [7], [2], [4], [13], [6], [11], [20], [10], [23], [8], [16], [1], [21], [12], [15], [18], [14], and [3] provide the notation and terminology for this paper.

1. Preliminaries and Equivalences of Inconsistency

For simplicity, we use the following convention: a denotes a set, X, Y denote subsets of CQC-WFF, k, m, n denote natural numbers, p, q denote elements of

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CQC-WFF, P denotes a k-ary predicate symbol, l_1 denotes a variables list of k, and f, g denote finite sequences of elements of CQC-WFF.

Let D be a non empty set and let X be a subset of 2^D. Then $\bigcup X$ is a subset of D.

In the sequel A is a non empty finite subset of $\mathbb{N}$.

The following two propositions are true:

(1) Let f be a function from n into A. Suppose there exists m such that $\operatorname{succ} m = n$ and f is one-to-one and $\operatorname{rng} f = A$ and for all n, m such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $n < m$ holds $f(n) \in f(m)$. Then $f(\bigcup n) = \bigcup \operatorname{rng} f$.

(2) $\bigcup A \in A$ and for every a such that $a \in A$ holds $a \in \bigcup A$ or $a = \bigcup A$.

Let A be a set. The functor $\min^* A$ yielding a natural number is defined by:

(Def. 1)(i) $\min^* A \in A$ and for every k such that $k \in A$ holds $\min^* A \leq k$ if A is a non empty subset of $\mathbb{N}$,

(ii) $\min^* A = 0$, otherwise.

In the sequel C denotes a non empty set.

Next we state the proposition

(3) Let f be a function from $\mathbb{N}$ into C and X be a finite set. Suppose for all n, m such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $n < m$ holds $f(n) \subseteq f(m)$ and $X \subseteq \bigcup \operatorname{rng} f$. Then there exists k such that $X \subseteq f(k)$.

Let us consider X, p. The predicate $X \vdash p$ is defined as follows:

(Def. 2) There exists f such that $\operatorname{rng} f \subseteq X$ and $\vdash f \frown \langle p \rangle$.

Let us consider X. We say that X is consistent if and only if:

(Def. 3) For every p holds $X \nvdash p$ or $X \nvdash \neg p$.

Let us consider X. We introduce X is inconsistent as an antonym of X is consistent.

Let f be a finite sequence of elements of CQC-WFF. We say that f is consistent if and only if:

(Def. 4) For every p holds $\nvdash f \frown \langle p \rangle$ or $\nvdash f \frown \langle \neg p \rangle$.

Let f be a finite sequence of elements of CQC-WFF. We introduce f is inconsistent as an antonym of f is consistent.

Next we state several propositions:

(4) If X is consistent and $\operatorname{rng} g \subseteq X$, then g is consistent.

(5) If $\vdash f \frown \langle p \rangle$, then $\vdash f \frown g \frown \langle p \rangle$.

(6) X is inconsistent iff for every p holds $X \vdash p$.

(7) If X is inconsistent, then there exists Y such that $Y \subseteq X$ and Y is finite and inconsistent.

(8) If $X \cup \{p\} \vdash q$, then there exists g such that $\operatorname{rng} g \subseteq X$ and $\vdash g \frown \langle p \rangle \frown \langle q \rangle$.

(9) $X \vdash p$ iff $X \cup \{\neg p\}$ is inconsistent.

(10) $X \vdash \neg p$ iff $X \cup \{p\}$ is inconsistent.

2. Unions of Consistent Sets

We now state the proposition

(11) Let f be a function from $\mathbb{N}$ into $2^{\text{CQC-WFF}}$. Suppose that for all n, m such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $n < m$ holds $f(n)$ is consistent and $f(n) \subseteq f(m)$. Then $\bigcup \operatorname{rng} f$ is consistent.

3. Construction of a Henkin Model

In the sequel A is a non empty set, v is an element of $\boldsymbol{V}(A)$, and J is an interpretation of A.

We now state two propositions:

(12) If X is inconsistent, then for all J, v holds $J, v \not\models X$.

(13) $\{\text{VERUM}\}$ is consistent.

Let us observe that there exists a subset of CQC-WFF which is consistent.

In the sequel C_1 denotes a consistent subset of CQC-WFF.

The non empty set HCar is defined by:

(Def. 5) $\text{HCar} = \text{BoundVar}$.

Let P be an element of PredSym and let l_1 be a variables list of $\text{Arity}(P)$. Then $P[l_1]$ is an element of CQC-WFF.

Let us consider C_1. An interpretation of HCar is said to be a Henkin interpretation of C_1 if it satisfies the condition (Def. 6).

(Def. 6) Let P be an element of PredSym and r be an element of Rel(HCar). Suppose $\text{it}(P) = r$. Let given a. Then $a \in r$ if and only if there exists a variables list l_1 of $\text{Arity}(P)$ such that $a = l_1$ and $C_1 \vdash P[l_1]$.

The element valH of $\boldsymbol{V}(\text{HCar})$ is defined as follows:

(Def. 7) $\text{valH} = \text{id}_{\text{BoundVar}}$.

4. Some Properties of the Henkin Model

In the sequel J_1 is a Henkin interpretation of C_1.

We now state four propositions:

(14) $\text{valH} * l_1 = l_1$.

(15) $\vdash f \frown \langle \text{VERUM} \rangle$.

(16) $J_1, \text{valH} \models \text{VERUM}$ iff $C_1 \vdash \text{VERUM}$.

(17) $J_1, \text{valH} \models P[l_1]$ iff $C_1 \vdash P[l_1]$.

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