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On the Free Subset Property at Singular Cardinals

Peter Koepke

Mathematisches Institut der Universität Freiburg, Abteilung für Mathematische Logik, Albertstrasse 23b, D-7800 Freiburg, Federal Republic of Germany

Abstract. We give a proof of

Theorem 1. *Let* κ *be the smallest cardinal such that the free subset property* $Fr_\omega(\kappa, \omega_1)$ *holds. Assume* κ *is singular. Then there is an inner model with* ω_1 *measurable cardinals.*

1. Introduction

The well-known notions of Ramsey and Erdös cardinals can be weakened in various ways, to yield e.g., Rowbottom and Jónsson cardinals, or cardinals having a free subset property. Usually our interest is in the smallest cardinal having such a property. Now whereas the original large cardinal notion implied that this smallest cardinal was at least strongly inaccessible, the weak versions do not rule out that their smallest instances are easily accessible. Often cardinals of the order of measurability suffice to force such properties for accessible cardinals, and some equiconsistencies have been proved. In this paper we show that the following theories are equiconsistent: "ZFC + the smallest κ such that $Fr_\omega(\kappa, \omega_1)$ is singular" and "ZFC + there are ω_1 measurable cardinals".

Let us define the *free subset property* $Fr_\mu(\kappa, \lambda)$. By a *structure* we understand a first order structure S which usually includes the $\in$-relation. The *cardinality* of S is the cardinality of the underlying set $|S|$, the *length* of S is the number of constants, functions, and relations of S. For $X \subseteq S$, $S[X]$ is the substructure of S generated from X by the constants and functions of S. $X \subseteq S$ is *free* in S, if for every $x \in X$, $x \notin S[X \setminus \{x\}]$. For cardinals κ, λ, μ, $Fr_\mu(\kappa, \lambda)$ denotes the property: every structure of cardinality $\geqq \kappa$ and length $\leqq \mu$ has a free subset of cardinality $\geqq \lambda$. Basic information on $Fr_\mu(\kappa, \lambda)$ is contained in Devlin [1] and Koepke [4]. In [4] we showed that if κ is minimal with $Fr_\omega(\kappa, \omega_1)$ then $\kappa \geqq \omega_{\omega_1}$, and $\mathrm{cof}(\kappa) = \omega_1$ or $\mathrm{cof}(\kappa) = \kappa$. Shelah [7] showed that one can force $Fr_\omega(\omega_{\omega_1}, \omega_1)$ starting from ω_1 measurable cardinals. Conversely we proved in [6] that $Fr_\omega(\omega_{\omega_1}, \omega_1)$ implies the existence of ω_1 measurable cardinals in an inner model. Here we strengthen this result to:

Theorem 1. *Let κ be the smallest cardinal such that $Fr_\omega(\kappa, \omega_1)$ holds. Let κ be singular. Then there is an inner model with ω_1 measurable cardinals.*

An inspection of the proof will yield the generalized

Theorem 2. *Let λ be an uncountable regular cardinal. Let κ be the least cardinal such that $Fr_\omega(\kappa, \lambda)$. Let κ be singular. Then there is an inner model with λ measurable cardinals.*

The proof of Theorem 1 is quite involved. We will first prove the existence of an inner model with one measurable cardinal from the assumptions, using the Dodd-Jensen *core model* K (see [3]). We will then indicate how we get the full result (ω_1 measurables) using the *short core models* of [6].

For the reader's convenience let us give a sketch of the proof: We assume that there is no inner model with a measurable cardinal and work for a contradiction. Let Z be an uncountable free subset for the structure K_{κ^+}. We consider transitive collapses $\bar{K}^Y$ of the substructures of K_{κ^+} generated by uncountable subsets Y of Z. By a suitable choice of Z we can ensure that all these $\bar{K}^Y$ are equal to one single structure $\bar{K}$.

The collapses embed canonically, and this allows to define measures on them. We show that for suitable Y, $\bar{K}^Y = \bar{K}$ is iterable by such a measure, and that the countable iterates are equal to $\bar{K}$. If $\bar{\kappa}$ is the ω_1-st iteration point of this iteration where $\bar{\kappa}$ is the largest cardinal in $\bar{K}$ we get a contradiction: $\bar{\kappa}$ must be regular in $\bar{K}$ because it is an iteration point, but it is singular in $\bar{K}$ using the Covering Theorem of Dodd and Jensen.

So the ω_1-st iteration point is always smaller than $\bar{\kappa}$, call it $\lambda^Y_{\omega_1}$. The ω_1-sequence of iteration points allows to define a mouse M^Y at $\lambda^Y_{\omega_1}$ which lies outside of $\bar{K}$. We can define M^Y for various Y's, so that the $\lambda^Y_{\omega_1}$ are cofinal in $\bar{\kappa}$ and the $\bar{K}^Y$ are all equal to some $\bar{K}$. The M^Y *descend* in the $\leq$-wellordering of mice, when their critical points increase towards $\bar{\kappa}$. So eventually these mice are all mouse-iterates of a single mouse M. The iteration points of M are cofinal in $\bar{\kappa}$, hence $\bar{\kappa}$ is an iteration point of M. M must lie outside $\bar{K}$, so $\bar{\kappa}$ is regular in $\bar{K}$. But this is a contradiction as above.

We should remark that the proof of iterability for sufficiently many $\bar{K}^Y$ combines ideas of Devlin and Paris [2] and of the proof of Kunen's result that a non-trivial elementary embedding $\pi: L \to_e L$ yields $O^\#$, as presented in [3, Sect. 12].

2. Getting One Measurable Cardinal

Assume that κ is minimal such that $Fr_\omega(\kappa, \omega_1)$, and assume that $\mathrm{cof}(\kappa) = \omega_1$ (By Koepke [4], we have either $\mathrm{cof}(\kappa) = \kappa$ or $\mathrm{cof}(\kappa) = \omega_1$). We will show in this chapter that there is an inner model with one measurable cardinal.

We proceed by contradiction and assume that there is no inner model with a measurable cardinal. Then by the Covering Theorem for K [3, 19.26],

(1) $\kappa^+ = (\kappa^+)^K$, and $K_{\kappa^+} \models \kappa$ is singular, where K_{κ^+} is $(H_{\kappa^+})^K$.

Let $\delta = \mathrm{cof}^K(\kappa)$. From now on we denote by K_{κ^+} the structure $\langle K_{\kappa^+}, \langle \alpha | \alpha \leqq \delta \rangle, \ldots \rangle$, where the α are constants, and ... stands for a countable set of Skolem functions for the structure K_{κ^+} without the added constants.

By Sect. 1 of Koepke [4] there exists a *good* free subset Z of K_{κ^+}, i.e.,

(2) Z is a cofinal subset of κ, $\mathrm{opt}(Z) = \omega_1$, and

$$\forall z \in Z \quad z \notin K_{\kappa^+}[z \cup (Z \setminus \{z\})].$$

So the elements of Z are also free relative to smaller ordinals. Note that every uncountable subset of Z also satisfies (2).

For uncountable $Y \subseteq Z$ define: $K^Y := K_{\kappa^+}[Y]$, $\sigma^Y : K^Y \cong \bar{K}^Y$, where $\bar{K}^Y$ is transitive. For uncountable $X \subseteq Y \subseteq Z$ define $\sigma^{XY} := \sigma^Y \circ (\sigma^X)^{-1} : \bar{K}^X \to_e \bar{K}^Y$; the subscript "$e$" signifies that the embedding is elementary; we also write $A <_e B$ if A is an elementary substructure of B. Every $\bar{K}^Y$ is a model of $ZFC^- + V = K$. The notion of "mouse" is absolute between $\bar{K}^Y$ and V, because $\omega_1 \subseteq \bar{K}^Y$. A proof of the following proposition is contained in the proof of [3, 14.19]:

(3) Let S, T be transitive models of $ZFC^- + V = K$. Let $\sigma : S \to_e T$, and $\omega_1 \subseteq S$. Then $S \subseteq T$.

We say that an uncountable $Y \subseteq Z$ is *cute* it for all uncountable $X \subseteq Y : \bar{K}^X = \bar{K}^Y$.

(4) There exists a cute $Y \subseteq Z$.

Proof. Assume not. There exists an ω-sequence $Z \supseteq Y_0 \supseteq Y_1 \supseteq \ldots$, such that Y_m is uncountable and such that $\bar{K}^{Y_m} \neq \bar{K}^{Y_{m+1}}$, for $m < \omega$. $\sigma^{Y_n Y_m} : \bar{K}^{Y_n} \to_e \bar{K}^{Y_m}$, for $m \leqq n < \omega$. So the ordinal height $On \cap \bar{K}^{Y_n}$ decreases monotonely with n growing. We can hence assume that $On \cap \bar{K}^{Y_m} = On \cap \bar{K}^{Y_n}$ for all $m, n < \omega$. (3) implies that $\bar{K}^{Y_n}$ is a proper subset of $\bar{K}^{Y_m}$ for $m < n < \omega$. For $m < \omega$ pick a mouse $M_m \in \bar{K}^{Y_m} \setminus \bar{K}^{Y_{m+1}}$. $M_{m+1} < M_m$ in the canonical order of mice defined in [3, 15.7]; it is easy to see that this order can be extended to the class of all mice. But $<$ is a well-ordering [3, 15.10], contradiction. QED (4)

By (4), we can assume that Z is cute. Set $\bar{K} := \bar{K}^Z$ and $\bar{\kappa} := \sigma^Z(\kappa)$. For every uncountable $Y \subseteq Z$, $(\sigma^Y)^{-1} : \bar{K} \to_e K_{\kappa^+}$ and $\sigma^Y(\kappa) = \bar{\kappa}$. For uncountable $X \subseteq Y \subseteq Z$, $\sigma^{XY} : \bar{K} \to_e \bar{K}$;

The following construction of an iteration of $\bar{K}$ is dependent on Z. Set $\sigma := \sigma^Z$. Let $\bar{Z} := \sigma'' Z$ and $\lambda_0 := \min(\bar{Z})$. Let $S := \bar{K}[\lambda_0 \cup (\bar{Z} \setminus \{\lambda_0\})]$, $\varrho : \bar{S} \cong S <_e \bar{K}$, $\bar{S}$ transitive.

(5) $\bar{S} = \bar{K}$, and $\varrho : \bar{K} \to_e \bar{K}$ has critical point λ_0.

Proof. $\lambda_0 \subseteq S$. $\lambda_0 \notin S$, since by (2):

$$\sigma^{-1}(\lambda_0) \notin K_{\kappa^+}[\sigma^{-1}(\lambda_0) \cup (Z \setminus \{\sigma^{-1}(\lambda_0)\})].$$

So λ_0 is the critical point of ϱ. We can define $\bar{\varrho} : \bar{K} \to_e \bar{S}$ by $\bar{\varrho} := \varrho^{-1} \circ \sigma^{X_0 Z}$, where $X_0 = Z \setminus \{\sigma^{-1}(\lambda_0)\}$. Then $\bar{K} \xrightarrow{\bar{\varrho}}_e \bar{S} \xrightarrow{\varrho}_e \bar{K}$, and by (3), $\bar{S} = \bar{K}$. QED (5)

Define $U_0 := \{x \in P(\lambda_0) \cap \bar{K} | \lambda_0 \in \varrho(x)\}$.

(6) $\langle \bar{K}_{\lambda_0^+}, U_0 \rangle$ is amenable, and $\langle \bar{K}_{\lambda_0^+}, U_0 \rangle \models U_0$ is a normal measure on λ_0, where $\bar{K}_{\lambda_0^+} := (H_{\lambda_0^+})^{\bar{K}}$.

Proof. Standard, see [3, 12.14]. QED (6).

In the following we will define an iteration $\langle \bar{K}, U_i \rangle_{i<\beta(Z)}$ of $\langle \bar{K}, U_0 \rangle$ with iteration maps $\langle \pi_{ij} \rangle_{i \leqq j < \beta(Z)}$ and critical points $\langle \lambda_i \rangle_{i<\beta(Z)}$. Simultaneously, we will define sequences $\langle X_i | i \leqq \beta(Z) \rangle$, $\langle Q_i | i \leqq \beta(Z) \rangle$, where each X_i is a subset of Z, and each Q_i is a set of subsets of Z. $\beta(Z)$ is an ordinal $\leqq \omega_1$ and will be determined in the construction; if $\beta(Z) = \omega_1$ then $\langle \bar{K}, U_0 \rangle$ is iterable, and if $\beta(Z) < \omega_1$, then the X_i, Q_i will be used to analyse the non-iterability of $\langle \bar{K}, U_0 \rangle$. We will ensure that the following property holds

(7) For $\beta \leqq \beta(Z)$, and for $i \leqq j < \beta$: $X_i \subseteq Z$, $\operatorname{card}(X_i) = \omega_1$, $X_i \supseteq X_j$, and for $v \in X_j$: $\pi_{0j} \circ \sigma(v) = \sigma(v)$.

We construct the iteration and the X_i, Q_i by recursion. Let $\beta \leqq \omega_1$, and assume that $\langle \bar{K}, U_i \rangle$, π_{ij}, λ_i, X_i, Q_i are defined for $i \leqq j < \beta$, obeying (7). We continue the construction at β according to various cases:

$\beta = 0$: $\langle \bar{K}, U_0 \rangle$, λ_0 are already fixed. Set $X_0 := Z \setminus \{\min(Z)\}$, and $Q_0 := \{\{\min(Z)\}\}$.
$\beta = 1$: Let $\pi_{01} : \bar{K} \to_{U_0} \tilde{K}$ be the ultrapower of $\bar{K}$ by U_0, where $\tilde{K}$ is transitive if it is well-founded.

Define an embedding $\tilde{\pi} : \tilde{K} \to \bar{K}$ by $\pi_{01}(f)(\lambda_0) \mapsto \varrho(f)(\lambda_0)$, for $f : \lambda_0 \to \bar{K}$, $f \in \bar{K}$. Łoś Theorem shows that $\tilde{\pi}$ is well-defined and elementary. Hence $\tilde{K}$ is transitive. Note that $\varrho = \tilde{\pi} \circ \pi_{01} : \bar{K} \xrightarrow{\pi_{01}}_e \tilde{K} \xrightarrow{\tilde{\pi}}_e \bar{K}$, and by (3), $\tilde{K} = \bar{K}$. Let $\lambda_1 := \pi_{01}(\lambda_0)$, and

$$U_1 := \bigcup \{\pi_{01}(x \cap U_0) | x \in \bar{K}_{\lambda_0^+}\}.$$

Then $\pi_{01} : \langle \bar{K}, U_0 \rangle \to \langle \bar{K}, U_1 \rangle$ is the one-step iteration of $\langle \bar{K}, U_0 \rangle$. Set $X_1 := \{v \in X_0 | \sigma^{X_0 Z}(\sigma(v)) = \sigma(v)\}$.

(8) If $v \in X_1$, then $\pi_{01}(\sigma(v)) = \sigma(v)$.

Proof. We had $\varrho = \tilde{\pi} \circ \pi_{01}$. In the proof of (5), we defined $\bar{\varrho} : \bar{K} \to_e \bar{K}$ such that $\varrho \circ \bar{\varrho} = \sigma^{X_0 Z}$. $\sigma^{X_0 Z} = \tilde{\pi} \circ \pi_{01} \circ \bar{\varrho}$, and so if $\sigma^{X_0 Z}(\sigma(v)) = \sigma(v)$, then $\pi_{01}(\sigma(v)) = \sigma(v)$. QED (8)

If $\operatorname{card}(X_1) = \omega_1$, set $Q_1 := \{X_0 \setminus X_1\}$ and continue. If $\operatorname{card}(X_1) < \omega_1$, set $Q_1 := \{X_0 \setminus X_1, X_1\}$, and finish the construction by setting $\beta(Z) := 1$. We note that

(9) If $Y \in Q_1$ has cardinality ω_1, then $Y = X_0 \setminus X_1$, and for $v \in Y$: $\sigma^Z(v) > \sigma^Y(v)$.

Proof. $\sigma^Z(v) = \sigma^{X_0 Z}(\sigma^{X_0}(v))$, by definition of $\sigma^{X_0 Z}$. $\sigma^Z(v) < \sigma^{X_0 Z}(\sigma^Z(v))$, since $v \notin X_1$. Hence $\sigma^Z(v) > \sigma^{X_0}(v) \geqq \sigma^Y(v)$, since $Y \subseteq X_0$. QED (9)

$\beta = \beta' + 1$, $\beta' \geqq 1$: Let $\pi_{\beta'\beta} : \bar{K} \to_{U_{\beta'}} \tilde{K}$ be the ultrapower of $\bar{K}$ by U_0 where $\tilde{K}$ is transitive if it is well-founded. Every element of $\tilde{K}$ is, in $\tilde{K}$, of the form $\pi_{0\beta}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, \lambda_{\beta'})$, where $f : \lambda_0^{n+1} \to \bar{K}$, $f \in \bar{K}$, $i(1) < \ldots < i(n) < \beta'$.

We want to establish a relation between such representations of elements of $\tilde{K}$, and elements of $\bar{K}$: Since $\pi_{01} : \bar{K} \to_{U_0} \bar{K}$, every element of $\bar{K}$ is of the form $\pi_{01}(g)(\lambda_0)$, where $g : \lambda_0 \to \bar{K}$, $g \in \bar{K}$. $\pi_{0\beta'} : \bar{K} \to \bar{K}$ is an iterated ultrapower, so every element of $\bar{K}$ is of the form

$$\pi_{0\beta'}(\pi_{01}(g)(\lambda_0))(\lambda_{i(1)}, \ldots, \lambda_{i(n)}),$$

where $g : \lambda_0 \to \bar{K}$, $g \in \bar{K}$, $\forall v < \lambda_0\ g(v) : v^n \to \bar{K}$, and $i(1) < \ldots < i(n) < \beta'$.

Now this can be rewritten as: $\pi_{0\beta'} \circ \pi_{01}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, \lambda_{\beta'})$, if we define $f: \lambda_0^{n+1} \to \bar{K}$ by:

$$f(x_1, \ldots, x_n, x_{n+1}) := g(x_{n+1})(x_1, \ldots, x_n),$$

if $x_1, \ldots, x_n < x_{n+1}$, and $f(x_1, \ldots, x_n, x_{n+1}) := 0$ else. These representations are homolog:

(10) Let φ be a Σ_0-formula, with one free variable for notational simplicity. Let $f: \lambda_0^{n+1} \to \bar{K}$, $f \in \bar{K}$, $i(1) < \ldots < i(n) < \beta'$. Then:

$$\tilde{K} \models \varphi(\pi_{0\beta}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, \lambda_{\beta'}))$$

iff

$$\bar{K} \models \varphi(\pi_{0\beta'} \circ \pi_{01}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, \lambda_{\beta'})).$$

Proof. We introduce a quantifier Q with the intension "there are measure one many":

$$\langle \bar{K}, U_i \rangle \models Qx\psi \quad \text{iff} \quad \{x < \lambda_i | \langle \bar{K}, U_i \rangle \models \psi(x)\} \in U_i.$$

Then:

$$\tilde{K} \models \varphi(\pi_{0\beta}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, \lambda_{\beta'}))$$

iff $\langle \bar{K}, U_{\beta'} \rangle \models Qx_{n+1}\varphi(\pi_{0\beta'}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, x_{n+1}))$

$$\vdots$$

iff $\langle \bar{K}, U_0 \rangle \models Qx_1 \ldots Qx_{n+1}\varphi(f(x_1, \ldots, x_n, x_{n+1}))$

We reduce the right hand side of (10) to the same form:

$$\bar{K} \models \varphi(\pi_{0\beta'} \circ \pi_{01}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, \lambda_{\beta'}))$$

iff $\langle \bar{K}, U_{i(n)} \rangle \models Qx_n\varphi(\pi_{0,i(n)} \circ \pi_{01}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n-1)}, x_n, \lambda_{i(n)}))$

$$\vdots$$

iff $\langle \bar{K}, U_0 \rangle \models Qx_1 \ldots Qx_n\varphi(\pi_{01}(f)(x_1, \ldots, x_n, \lambda_0))$

iff $\langle \bar{K}, U_0 \rangle \models Qx_1 \ldots Qx_nQx_{n+1}\varphi(f(x_1, \ldots, x_n, x_{n+1}))$,

because

$$\bar{K} \models \varphi(\pi_{01}(f)(x_1, \ldots, x_n, \lambda_0)) \quad \text{iff} \quad \langle \bar{K}, U_0 \rangle \models Qx_{n+1}\varphi(f(x_1, \ldots, x_n, x_{n+1})),$$

for $x_1, \ldots, x_n < \lambda_0$. QED (10)

By (10), the assignment

$$\pi_{0\beta}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, \lambda_{\beta'}) \mapsto \pi_{0\beta'} \circ \pi_{01}(f)(\lambda_{i(1)}, \ldots, \lambda_{i(n)}, \lambda_{\beta'})$$

defines a Σ_0-elementary embedding from $\tilde{K}$ into $\bar{K}$. By the remarks preceding (10), this embedding is onto. Hence:

(11) $$\tilde{K} = \bar{K}, \quad \text{and} \quad \pi_{0\beta} = \pi_{0\beta'} \circ \pi_{01}.$$

Set $X_\beta := X_{\beta'}$, $Q_\beta := \emptyset$, and continue the construction.

(12) If $v \in X_\beta$, then $\pi_{0\beta}(\sigma(v)) = \sigma(v)$.

Proof. $\pi_{0\beta}(\sigma(v))=\pi_{0\beta'}(\pi_{01}(\sigma(v)))=\pi_{0\beta'}(\sigma(v))=\sigma(v)$. QED (12)

$\mathrm{Lim}(\beta)$, and $\beta<\omega_1$: Set

$$X:=\bigcap_{i<\beta} X_i.$$

If $\mathrm{card}(X)<\omega_1$, set $X_\beta:=\emptyset$, $Q_\beta:=\{X\}$, and finish the construction by setting $\beta(Z):=\beta$.

Now assume that $\mathrm{card}(X)=\omega_1$. We will show that in this case $\langle\langle\bar{K},U_i\rangle,\pi_{ij}\rangle_{i\leqq j<\beta}$ has a limit $\langle\bar{K},U_\beta\rangle$.

Set $C:=\{x\in\bar{K}|\pi_{0i}(x)=x$, for all $i<\beta\}$.

(13) $$\sigma''X\subseteqq C,\quad \lambda_0\subseteqq C,\quad \text{and}\quad C<_e\bar{K}.$$

Proof. Obvious. QED (13)

For $i<\beta$ set $C_i:=\bar{K}[\{\lambda_j|j<i\}\cup C]$. Let $\tilde{\pi}_i:\tilde{K}_i\cong C_i$, $\tilde{K}_i$ transitive.

(14) $$\tilde{K}_i=\bar{K},\quad \text{for}\quad i<\beta.$$

Proof. Define $\pi_i':\bar{K}\to_e\tilde{K}_i$ by: $\pi_i':=\tilde{\pi}_i^{-1}\circ\sigma\circ(\sigma^X)^{-1}$. Then $\bar{K}\xrightarrow{\pi_i'}_e\tilde{K}_i\xrightarrow{\tilde{\pi}_i}_e\bar{K}$, and the result follows from (3). QED (14)

For $i\leqq j<\beta$ define $\tilde{\pi}_{ij}:=\tilde{\pi}_j^{-1}\circ\tilde{\pi}_i:\bar{K}\to_e\bar{K}$.

We will show that $\tilde{\pi}_{ij}=\pi_{ij}$.

(15) For $a\in P(\lambda_0)\cap\bar{K}$, $\pi_{01}(a)=\tilde{\pi}_0(a)\cap\lambda_i$.

Proof. $a=\tilde{\pi}_0(a)\cap\lambda_0$, since $\tilde{\pi}_0\restriction\lambda_0=\mathrm{id}$. $\pi_{0i}(a)=\pi_{0i}(\tilde{\pi}_0(a)\cap\lambda_0)=\tilde{\pi}_0(a)\cap\lambda_i$, since $\tilde{\pi}_0(a)\in C$, and $\pi_{0i}\restriction C=id$. QED (15)

(16) For $i<\beta$, $\lambda_i\subseteqq C_i$.

Proof. Let $v<\lambda_i$, $v=\pi_{0i}(f)(\lambda_{i(1)},\ldots,\lambda_{i(n)})$, $f\in\bar{K}$, $f:\lambda_0^n\to\bar{K}$, $i(1)<\ldots<i(n)<i$. We can assume that $f:\lambda_0^n\to\lambda_0$. By (15), $\pi_{0i}(f)=\tilde{\pi}_0(f)\restriction\lambda_i^n$. So

$$v=\tilde{\pi}_0(f)(\lambda_{i(1)},\ldots,\lambda_{i(n)})\in C_i.\quad \text{QED (16)}$$

(17) $$x\in C_i\to\pi_{jk}(x)=x,\quad \text{for}\quad i\leqq j\leqq k<\beta.$$

Proof. x is definable in $\bar{K}$ from some $\hat{x}\in C$ and $\lambda_{i(1)},\ldots,\lambda_{i(n)}$, $i(1)<\ldots<i(n)<\beta$. Now π_{jk} maps $\hat{x}$, $\lambda_{i(1)},\ldots,\lambda_{i(n)}$ identically. QED (17)

(18) $$C_i\cap\lambda_j=\lambda_i,\quad \text{for}\quad i\leqq j<\beta.$$

Proof. $\supseteqq$ *by* (16). $\subseteqq$: Assume $\gamma\in C_i\cap\lambda_j$ and $\gamma\geqq\lambda_i$. By (17), $\pi_{ij}(\gamma)=\gamma$, but $\pi_{ij}(\gamma)\geqq\pi(\lambda_i)=\lambda_j$. Contradiction. QED (18)

(19) $$\tilde{\pi}_{ij}(\lambda_i)=\lambda_j,\quad \text{for}\quad i\leqq j<\beta.$$

Proof. $\tilde{\pi}_{ij}(\lambda_i)=\tilde{\pi}_j^{-1}\circ\tilde{\pi}_i(\lambda_i)=\mathrm{otp}(C_j\cap\tilde{\pi}_i(\lambda_i))\geqq\mathrm{otp}(C_j\cap\lambda_j)$, since

$$\tilde{\pi}_i(\lambda_i)=\pi_{ij}(\tilde{\pi}_i(\lambda_i))\geqq\pi_{ij}(\lambda_i)=\lambda_j.$$

Now suppose that $\tilde{\pi}_{ij}(\lambda_i) > \lambda_j$, hence $\mathrm{otp}(C_j \cap \tilde{\pi}_i(\lambda_i)) > \mathrm{otp}(C_j \cap \lambda_j)$. There is

$$t(\lambda_{i(1)}, \ldots, \lambda_{i(n)}) \in C_j ,$$

such that $t \in C$, $i(1) < \ldots < i(n) < j$, and $\lambda_j \leqq t(\lambda_{i(1)}, \ldots, \lambda_{i(n)}) < \tilde{\pi}_i(\lambda_i)$.

$$\bar{K} \models \exists \xi_1, \ldots, \xi_n < \lambda_j \cdot \lambda_j \leqq t(\xi_1, \ldots, \xi_n) < \tilde{\pi}_i(\lambda_i) .$$

Applying π_{ij}^{-1}:

$$\bar{K} \models \exists \xi_1, \ldots, \xi_n < \lambda_i \cdot \lambda_i \leqq t(\xi_1, \ldots, \xi_n) < \tilde{\pi}_i(\lambda_i) .$$

Such a $t(\xi_1, \ldots, \xi_n)$ would be in C_i, by (16), but, again by (16), $C_i \cap \tilde{\pi}_i(\lambda_i) = \lambda_i$. Contradiction. QED (19)

(20) If $a \in P(\lambda_i) \cap \bar{K}$ then $\pi_{ij}(a) = \tilde{\pi}_{ij}(a)$.

Proof. $a = \tilde{\pi}_i(a) \cap \lambda_i$. So $\pi_{ij}(a) = \pi_{ij}(\tilde{\pi}_i(a) \cap \lambda_i) = \tilde{\pi}_i(a) \cap \lambda_j = (\tilde{\pi}_j^{-1} \circ \tilde{\pi}_i(a)) \cap \lambda_j$ [by (16)] $= \tilde{\pi}_{ij}(a) \cap \lambda_j = \tilde{\pi}_{ij}(a)$, by (19). QED (20)

We compare the systems $\langle \pi_{ij} \rangle$ and $\langle \tilde{\pi}_{ij} \rangle$: Recursively, define functions $\sigma_i : \bar{K} \to \bar{K}$ by: $\sigma_0 := id \restriction \bar{K}$;

$$\sigma_{i+1}(\pi_{ii+1}(f)(\lambda_i)) := \tilde{\pi}_{ii+1}(f)(\lambda_i), \quad \text{for} \quad f \in \bar{K}, f : \lambda_i \to \bar{K};$$

$\sigma_l(\pi_{il}(x)) := \tilde{\pi}_{il}(x)$, for $x \in \bar{K}$, $i < l$, where l is a limit ordinal $< \beta$. We verify inductively:

(21) Each σ_i is well-defined and is the identity on $\bar{K}$.

The claim is trivial for $i = 0$.

Let $i = j + 1$, and assume (21) holds for j.

(22) Let φ be a formula, $\pi_{ji}(f_1)(\lambda_j), \ldots, \pi_{ji}(f_n)(\lambda_j) \in \bar{K}$, and $f_1, \ldots, f_n \in \bar{K}$. Then

$$\bar{K} \models \varphi(\pi_{ji}(f_1)(\lambda_j), \ldots, \pi_{ji}(f_n)(\lambda_j))$$

iff $\bar{K} \models \varphi(\tilde{\pi}_{ji}(f_1)(\lambda_j), \ldots, \tilde{\pi}_{ji}(f_n)(\lambda_j))$.

Proof. $\bar{K} \models \varphi(\pi_{ji}(f_1)(\lambda_j), \ldots, \pi_{ji}(f_n)(\lambda_j))$

iff $\lambda_j \in \pi_{ji}(\{\nu < \lambda_j | \varphi(f_1(\nu), \ldots, f_n(\nu))\})$

iff $\lambda_j \in \tilde{\pi}_{ji}(\{\nu < \lambda_j | \varphi(f_1(\nu), \ldots, f_n(\nu))\})$, by (20),

iff $\bar{K} \models \varphi(\tilde{\pi}_{ji}(f_1)(\lambda_j), \ldots, \tilde{\pi}_{ji}(f_n)(\lambda_j))$. QED (22)

So σ_i is well-defined and elementary. To conclude the case $i = j + 1$, it suffices to show:

(23) σ_i is onto.

Proof. Let $x \in \bar{K}$. $x = \tilde{\pi}_i^{-1} \circ \tilde{\pi}_i(x) = \tilde{\pi}_i^{-1}(t(\lambda_j))$, for some $t \in C_j$. So

$$\begin{aligned} x &= (\tilde{\pi}_i^{-1}(t))(\tilde{\pi}_i^{-1}(\lambda_j)) = (\tilde{\pi}_i^{-1}(t))(\lambda_j) \\ &= (\tilde{\pi}_i^{-1}\tilde{\pi}_j(\bar{t}))(\lambda_j), \quad \text{for some } \bar{t} \in \bar{K}, \text{ since } C_j = \mathrm{range}(\tilde{\pi}_j), \\ &= \tilde{\pi}_{ji}(\bar{t})(\lambda_j) = \sigma_i(\pi_{ji}(\bar{t})(\lambda_j)) \in \mathrm{range}(\sigma_i). \quad \text{QED (23)} \end{aligned}$$

Finally assume Lim(i), and that (21) holds for $j<i$. $\langle\pi_{ji}\rangle_{j<i}$ is the limit of $\langle\pi_{jk}\rangle_{j\leqq k<i}$, and $\langle\tilde{\pi}_{ji}\rangle_{j<i}$ is the limit of $\langle\tilde{\pi}_{jk}\rangle_{j\leqq k<i}$. By inductive hypothesis, the systems $\langle\pi_{jk}\rangle_{j\leqq k<i}$ and $\langle\tilde{\pi}_{jk}\rangle_{j\leqq k<i}$ are equal. Hence $\pi_{ji}=\tilde{\pi}_{ji}$ for $j<i$.

So σ_i is well-defined and is the identity on $\bar{K}$, and we have verified:

(24) $$\pi_{ij}=\tilde{\pi}_{ij}, \quad \text{for} \quad i\leqq j<\beta.$$

(25) The system $\langle\bar{K},\langle\pi_{ij}\rangle\rangle_{i\leqq j<\beta}$ has a well-founded direct limit $\langle\tilde{K},\langle\pi_{i\beta}\rangle\rangle_{i<\beta}$, and there is a map $\tilde{\pi}:\tilde{K}\to_e\bar{K}$.

Proof. Let $\langle\tilde{K},\langle\pi_{i\beta}\rangle\rangle$ be a direct limit of $\langle\bar{K},\langle\pi_{ij}\rangle\rangle$, which is supposed to be transitive if it is well-founded. Define $\tilde{\pi}:\tilde{K}\to_e\bar{K}$ by $\tilde{\pi}(\pi_{i\beta}(x)):=\tilde{\pi}_i(x)$. Since $\tilde{\pi}_i=\tilde{\pi}_j\circ\tilde{\pi}_{ij}=\tilde{\pi}_j\circ\pi_{ij}$, for $i\leqq j<\beta$, $\tilde{\pi}$ is well-defined and elementary. Hence $\tilde{K}$ is transitive. QED (25)

(26) $$\tilde{K}=\bar{K}.$$

Proof. $\bar{K}\xrightarrow{\tilde{\pi}_{i\beta}}_e\tilde{K}\xrightarrow{\tilde{\pi}}_e\bar{K}$. Use (3). QED (26)

Set $U_\beta:=\bigcup\{\pi_{0\beta}(x\cap U_0)|x\in\bar{K}_{\lambda_\delta^+}\}$. $\langle\langle\bar{K},U_\beta\rangle,\pi_{i\beta}\rangle_{i<\beta}$ is the limit of $\langle\langle\bar{K},U_i\rangle,\pi_{ij}\rangle_{i\leqq j<\beta}$. We now have to check whether there are enough fixed points for $\pi_{0\beta}$ to keep the construction going.

Set $X_\beta:=\{v\in X|\sigma^{XZ}(\sigma(v))=\sigma(v)\}$.

(27) If $v\in X_\beta$, then $\pi_{0\beta}(\sigma(v))=\sigma(v)$.

Proof. By the proof of (25), there is $\tilde{\pi}$ such that $\tilde{\pi}\circ\pi_{0\beta}=\tilde{\pi}_0$. In the proof of (14) we defined $\pi_0':\bar{K}\to_e\bar{K}$ by
$$\pi_0'=\tilde{\pi}_0^{-1}\circ\sigma^Z\circ(\sigma^X)^{-1}=\tilde{\pi}_0^{-1}\circ\sigma^{XZ}.$$
$\sigma^{XZ}=\tilde{\pi}_0\circ\pi_0'=\tilde{\pi}\circ\pi_{0\beta}\circ\pi_0'$. So if $v\in X_\beta$, $\sigma(v)$ is a fixed point of σ^{XZ}, and therefore $\sigma(v)$ is a fixed point of $\pi_{0\beta}$. QED (27)

We distinguish two cases:

If $\mathrm{card}(X_\beta)=\omega_1$, set $Q_\beta:=\{X\backslash X_\beta\}$, and continue the construction.

If $\mathrm{card}(X_\beta)<\omega_1$, set $Q_\beta:=\{X\backslash X_\beta, X_\beta\}$, and finish the construction by setting $\beta(Z):=\beta$.

In either case we note:

(28) If $Y\in Q_\beta$ has cardinality ω_1, then $Y=X\backslash X_\beta$, and for
$$v\in Y:\sigma^Z(v)>\sigma^Y(v).$$

Proof. $\sigma^Z(v)=\sigma^{XZ}(\sigma^X(v))$, by definition of σ^{XZ}. $\sigma^Z(v)<\sigma^{XZ}(\sigma^Z(v))$, since $v\notin X_\beta$. Hence $\sigma^Z(v)>\sigma^X(v)\geqq\sigma^Y(v)$, since $Y=X\backslash X_\beta\subseteqq X$. QED (28)

Finally, we consider the case:

$\beta=\omega_1$: Set $\beta(Z):=\omega_1$, $X_\beta:=\emptyset$, $Q_\beta:=\emptyset$. Then the iterate $\langle\bar{K},U_i\rangle$ exists for all $i<\omega_1$, and using the ideas of [3, Lemma 8.6], we see

(29) If $\beta(Z)=\omega_1$, then $\langle\bar{K},U_0\rangle$ is iterable.

This concludes the construction of our system. We note the following properties

(30) If $\beta(Z)<\omega_1$, then $\bigcup\{Q_\beta|\beta\leqq\beta(Z)\}$ is a partition of Z into countably many subsets.

Proof. Obvious from the construction. QED (30)

(31) If $Y\in\bigcup\{Q_\beta|\beta\leqq\beta(Z)\}$ has cardinality ω_1, then for

$$v\in Y: \sigma^Z(v)>\sigma^Y(v).$$

Proof. By (9) and (28). QED (31)

The above construction was dependent on the cute set Z, and since we shall have to vary Z, we now write $\pi_{ij}^Z, U_i^Z, \lambda_i^Z, Q_\beta^Z, \ldots$ instead of $\pi_{ij}, U_i, \lambda_i, Q_\beta, \ldots$.

(32) There is a cute set $X\subsetneqq Z$ such that $\beta(X)=\omega_1$.

Proof. Assume that $\beta(X)<\omega_1$ for all uncountable $X\subsetneqq Z$. We build a tree T of subsets of Z. T has height ω. $T=\bigcup_{n<\omega} T_n$, where T_n denotes the n-th level of T.

Set $T_0:=\{Z\}$. So Z is the root of T. If $Y\in T_n$ has cardinality $<\omega_1$, the unique successor of Y at level T_{n+1} is Y again. If $Y\in T_n$ has cardinality ω_1, then the successors of Y at level T_{n+1} are all the elements of $\bigcup\{Q_\beta^Y|\beta\leqq\beta(Y)\}$. Since $\beta(Y)<\omega_1$, the immediate successors of Y at level T_{n+1} partition Y into countably many pieces. Every level T_n yields a partition of Z into pairwise disjoint sets: $Z=\bigcup T_n$.

The ordering of T coincides with reverse inclusion. T has countably many nodes. So we can pick $v\in Z$, so that for all $n<\omega$, v is a member of an *uncountable* element of T_n. Say $v\in Y_n\in T_n$, $\text{card}(Y_n)=\omega_1$ $(n<\omega)$. Then $Y_0\supsetneqq Y_1\supsetneqq Y_2\supsetneqq\ldots$, and using (31) we get:

$$\sigma^{Y_0}(v)>\sigma^{Y_1}(v)>\sigma^{Y_2}(v)>\ldots.$$

Contradiction. QED (32)

Because Z was an arbitrary cute set (32) actually proves:

(33) For every uncountable $Y\subsetneqq Z$ there exists an uncountable $X\subsetneqq Y$ such that $\beta(X)=\omega_1$.

We conclude the proof of Theorem 1 according to two cases:

Case 1. There exists an uncountable $X\subsetneqq Z$ such that $\beta(X)=\omega_1$ and $\{\lambda_i^X|i<\omega_1\}$ is cofinal in $\bar\kappa=\sigma^Z(\kappa)$.

Let $\langle\langle \bar K_i^X, U_i^X\rangle, \pi_{ij}^X\rangle_{i\leqq j\in On}$ with iteration points λ_i^X be the iteration of $\langle \bar K, U_0^X\rangle$. Then $\lambda_{\omega_1}^X=\bar\kappa$.

(34) $\bar K_{\omega_1}^X\models\bar\kappa$ is singular.

Proof. By (1), $\bar K\models\bar\kappa$ is singular. $\pi_{0\omega_1}^X:\bar K\to_e \bar K_{\omega_1}^X$, and by (3), $\bar K_{\omega_1}^X\supsetneqq\bar K$. Hence $\bar K_{\omega_1}^X\models\bar\kappa$ is singular. QED (34)

But this yields a *contradiction* since $\langle\bar K_{\omega_1}^X, U_{\omega_1}^X\rangle\models U_{\omega_1}^X$ is a measure on $\bar\kappa$, implying that $\bar K_{\omega_1}^X\models\bar\kappa$ is regular. This finishes the proof of Theorem 1 in Case 1.

Case 2. If $X \subseteqq Z$ is cute and $\beta(X)=\omega_1$, then $\{\lambda_i^X | i<\omega_1\}$ is bounded below $\bar{\kappa}$.

Let $X \subseteqq Z$ be uncountable with $\beta(X)=\omega_1$. Let $\langle\langle \bar{K}_i^X, U_i^X\rangle, \pi_{ij}^X\rangle_{i \leqq j \in On}$ with iteration points λ_i^X be the iteration of $\langle \bar{K}, U_0^X\rangle$.

We want to associate with X a mouse M^X at $\lambda_{\omega_1}^X$, which is not an element of $\bar{K}$. Set $N^X := J_\gamma[U_{\omega_1}^X]$, where γ is maximal such that $J_\gamma[U_{\omega_1}^X] \cap P(\lambda_{\omega_1}^X) \subseteqq \bar{K}_{\omega_1}^X$. γ exists because otherwise $L[U_{\omega_1}^X]$ would be an inner model with a measurable cardinal contradicting our initial assumption.

(35) $$\gamma \geqq \lambda_{\omega_1}^X + 1 .$$

Proof. Set $\lambda := \lambda_{\omega_1}^X$, $\tilde{K} := \bar{K}_{\omega_1}^X$. Then $J_{\lambda+1}[U_{\omega_1}^X] \subseteqq \bar{K}_{\omega_1}^X$, because $\langle H_{\lambda^+}^{\tilde{K}}, U_{\omega_1}^X\rangle$ is amenable. QED (35)

We distinguish two cases:

Case I. $P(\lambda_{\omega_1}^X) \cap N^X \subseteqq \bar{K}$.
Then set $M^X := N^X$, and say that M^X is of *type I*.

Case II. $P(\lambda_{\omega_1}^X) \cap N^X \nsubseteqq \bar{K}$.
Then set $M^X := J_{\eta+1}[U_{\omega_1}^X]$, where η is maximal such that $P(\lambda_{\omega_1}^X) \cap J_\eta[U_{\omega_1}^X] \subseteqq \bar{K}$. We say that this M^X is of *type II*.

(36) M^X is a mouse at $\lambda_{\omega_1}^X < \bar{\kappa}$.

Proof. $N^X \models U_{\omega_1}^X$ is a measure at $\lambda_{\omega_1}^X$, and $U_{\omega_1}^X$ is countably complete. $M^X \subseteqq N^X$ and $On \cap M^X > \lambda_{\omega_1}^X$. So $M^X \models U_{\omega_1}^X$ is a measure at $\lambda_{\omega_1}^X$. If M^X is of type I, $\Sigma_\omega(M^X) \cap P(\lambda_{\omega_1}^X) \nsubseteqq M^X$, and so some projectum of M^X drops to a point $\leqq \lambda_{\omega_1}^X$.

If M^X is of type II, then some projectum of $J_\eta[U_{\omega_1}^X]$, η as in the definition of M^X, drops to an ordinal $\leqq \lambda_{\omega_1}^X$. But then the first projectum $\varrho_{M^X}^1$ of M^X is $\leqq \lambda_{\omega_1}^X$. So in both cases, M^X is a mouse. QED (36)

(37) $$M^X \notin \bar{K} .$$

Proof. Because M^X contains or allows to define over it a subset of $\lambda_{\omega_1}^X$ which is not in $\bar{K}$. QED (37)

For X as above set $\lambda^X := \lambda_{\omega_1}^X$. We can find such λ^X cofinally in $\bar{\kappa}$:

(38) Let $\xi < \bar{\kappa}$. Then there exists an uncountable $X \subseteqq Z$ such that $\beta(X)=\omega_1$ and $\xi < \lambda^X < \bar{\kappa}$.

Proof. In $\bar{K}$, let f be the $<_{\bar{K}}$-least function such that $f: \mathrm{cof}(\bar{\kappa}) \to \bar{\kappa}$ cofinally. Choose i such that $f(i) > \xi$, and let

$$Y := \{\nu \in Z | \sigma^Z(\nu) > f(i)\} .$$

By (33) choose an uncountable $X \subseteqq Y$ such that $\beta(X)=\omega_1$. For $\nu \in X$,

$$\sigma^{XZ}(\sigma^X(\nu)) = \sigma^Z(\nu) > f(i) = \sigma^{XZ}(f(i)),$$

since i is a constant of $\bar{K}$; hence $\sigma^X(\nu) > f(i)$.

So $\lambda_0^X = \sigma^X(\min(X)) > f(i)$, and $\lambda^X > \lambda_0^X > f(i) > \xi$. QED (38)

(39) Let M^X, M^Y be of type I and $\lambda^X < \lambda^Y$. Then $M^X \geqq M^Y$, where $\leqq$ denotes the canonical well-ordering of mice [see the proof of (4)].

Proof. Assume $M^X < M^Y$ instead. There are mouse-iterates $\tilde{M}^X$, $\tilde{M}^Y$ of M^X, M^Y respectively such that $\tilde{M}^X \in \tilde{M}^Y$. Over $\tilde{M}^X$ we can define a subset $c \subseteq \lambda^X$ which codes M^X. $c \in \tilde{M}^Y$, and so $c \in M^Y$. Then $c \in \bar{K}$ and since we can decode c in $\bar{K}$, $M^X \in \bar{K}$. This contradicts (37). QED (39)

(40) Let M^X, M^Y be of type II and $\lambda^X < \lambda^Y$. Then $M^X \geqq M^Y$.

Proof. Assume $M^X < M^Y$ instead; let $\tilde{M}^X, \tilde{M}^Y$ be mouse-iterates of M^X, M^Y such that $\tilde{M}^X \in \tilde{M}^Y$. Let $M^Y = J_{\eta+1}[U]$ and $\tilde{M}^Y = J_{\tilde{\eta}+1}[\tilde{U}]$. $\tilde{M}^X$ contains a subset $c \subseteq \lambda^X$ such that $c \notin \bar{K}$. $c \in \tilde{M}^X \subseteq J_{\tilde{\eta}}[\tilde{U}]$. The iteration map from M^Y to $\tilde{M}^Y$ maps c identically (since $\lambda^Y > \lambda^X$), and maps η to $\tilde{\eta}$. Then $c \in J_\eta[U]$. By the definition of type II mice, $c \in \bar{K}$. Contradiction. QED (40)

Now by (38), we choose uncountable $X_i \subseteq Z$, for $i < \omega_1$, such that:

(41) $$\beta(X_i) = \omega_1;$$

(42) $$i < j < \omega_1 \rightarrow \lambda^{X_i} < \lambda^{X_j} < \bar{\kappa};$$

and

(43) $\{\lambda^{X_i} | i < \omega_1\}$ is cofinal in $\bar{\kappa}$.

We can further assume that the mice M^{X_i} are all of type I or all of type II. By (39) or (40) this implies:

(44) $$i < j < \omega_1 \rightarrow M^{X_i} \geqq M^{X_j}.$$

Since the ordering $\leqq$ of mice is well-founded, we can assume that

(45) $M^{X_i} \sim M^{X_j}$, for $i, j < \omega_1$ (write $N \sim N'$ for $N \leqq N'$ and $N' \leqq N$).

Then, using [3, 10.16]:

(46) M^{X_i} is a mouse-iterate of M^{X_0}, for $i < \omega_1$.

Set $M := M^{X_0}$. λ^{X_i} is the measurable of M^{X_i}, and therefore every λ^{X_i} is an iteration point of M. Since the λ^{X_i} are cofinal in $\bar{\kappa}$:

(47) $\bar{\kappa}$ is an iteration point of M in the mouse-iteration of M.

(48) $\bar{K} \models \bar{\kappa}$ is singular, by (1).

Let $N \in \bar{K}$ be a mouse such that $N \models \bar{\kappa}$ is singular, and such that the measurable of N is $> \bar{\kappa}$. Let $\tilde{M}, \tilde{N}$ be comparable mouse-iterates of M, N respectively. If $\tilde{N} \subseteq \tilde{M}$, then $\tilde{M} \models \bar{\kappa}$ is singular, although $\bar{\kappa}$ is an iteration point of M. So $\tilde{M} \in \tilde{N}$, and there is $c \in P(\bar{\kappa}) \cap \tilde{N}$, which codes M. $c \in N \in \bar{K}$, and, decoding c in $\bar{K}$, $M \in \bar{K}$. But this *contradicts* (37).

This concludes the proof of Theorem 1, as far as the existence of an inner model with *one* measurable cardinal is concerned. QED

3. How to Get ω_1 Measurable Cardinals

To derive the full result, i.e., the existence of ω_1 measurable cardinals in some inner model under the assumptions of Theorem 1, one uses the family of *short core* *models* as presented in [6]. The argument of Sect. 2 can be adapted to these larger core models and we indicate some of the changes necessary. The fine structure arguments used to prove facts (39) and (40) above have to be replaced by fine structure results developed in [5]. This means that we have to be very vague.

Again our proof proceeds by contradiction. Assume κ is minimal with $Fr_\omega(\kappa, \omega_1)$, and $\mathrm{cof}(\kappa)=\omega_1$, and assume there is no inner model with ω_1 measurable cardinals. By [6, 2.14], this implies $\neg O^{\mathrm{long}}$. So the fundamental properties of short core models hold. Let $K[U_{\mathrm{can}}]$ be the *canonical core model* [6, 3.15]. By the *covering theorem* [6, 3.19],

$$(1') \qquad \kappa^+ = (\kappa^+)^{K[U_{\mathrm{can}}]}.$$

$\mathrm{dom}(U_{\mathrm{can}})$ is countable, because otherwise $K[U_{\mathrm{can}}]$ would be an inner model of uncountably many measurable cardinals. So for any *Prikry system* C for $K[U_{\mathrm{can}}]$, the collection $\tilde{C}\cap\kappa$ of "Prikry points" $<\kappa$ in C is bounded below κ (see [6, 3.22]). By the *covering theorem with Prikry systems* [6, 3.23],

(1″) $K[U_{\mathrm{can}}] \models \kappa$ is singular.

So we have established the analogue of (1) of Sect. 2. Set $F := U_{\mathrm{can}} \upharpoonright \kappa$, and

$$\delta := \max(\mathrm{cof}^{K[U_{\mathrm{can}}]}(\kappa), \sup \mathrm{dom}(F)) < \kappa.$$

Let us denote by K_{κ^+} the structure $\langle (H_{\kappa^+})^{K[F]}, F, \langle \alpha | \alpha \leqq \delta \rangle, \ldots \rangle$, where F and the α are constants, and ... stands for a countable collection of Skolem functions for the structure K_{κ^+} without constants. For short core models over F property (3) holds in the form:

(3′) Let S, T be transitive models of $\mathrm{ZFC}^- + V = K[F]$, where $F \in S, T$. Let $\sigma: S \to_e T$, $\omega_1 \subseteq S$, such that $\sigma \upharpoonright (\sup \mathrm{dom}(F)+1) = id$. Then $S \subseteq T$.

With this, the arguments of Sect. 2 go through unchanged up to the consideration of

Case 2. If $X \subseteq Z$ is cute, and $\beta(X) = \omega_1$, then $\{\lambda_i^X | i < \omega_1\}$ is bounded below $\bar{\kappa}$.

Let $X \subseteq Z$ be uncountable with $\beta(X) = \omega_1$. Let $\langle\langle \bar{K}_i^X, U_i^X \rangle, \pi_{ij}^X \rangle_{i \leqq j \in On}$ with iteration points λ_i^X be the iteration of $\langle \bar{K}, U_0^X \rangle$. We determine a mouse M^X over F which is not an element of $\bar{K}$: Set $\lambda := \lambda_{\omega_1}^X$, $\tilde{K} := \bar{K}_{\omega_1}^X$. Let F' be the predicate with $\mathrm{dom}(F') = \mathrm{dom}(F) \cup \{\lambda\}$ such that $F' \upharpoonright \mathrm{dom}(F) = F$ and $F'_\lambda = U_{\omega_1}^X$.
By the definition of U_{can}, F' is not strong and there exists an iterable premouse $P = J_\alpha[G, F']$ over F' such that $P \models$ "F' is not a sequence of measures". We may assume that the predicate G is countably complete. Set $N := J_\gamma[G, F']$ where $\gamma < \alpha$ is maximal such that $J_\gamma[G, F'] \cap P(\lambda) \subseteq \tilde{K}$. We distinguish two cases:

Case I. $P(\lambda) \cap N \subseteq \tilde{K}$.

Then set $M := N$, and say that M is of *type I*.

Case II. $P(\lambda) \cap N \nsubseteq \tilde{K}$.

Then set $M := J_{\eta+1}[G, F']$, where η is maximal such that $P(\lambda) \cap J_\eta[G, F'] \subseteqq \bar{K}$, and say that M is of *type II*.

(36′) M can be viewed as a mouse over F, and then $\lambda = \min \operatorname{meas}(M)$.

This is the place where finestructure comes into play, and we become very sketchy. Basically, things behave as in Sect. 2 after some rather difficult definability and iterability questions are dealt with.

Now let M^X be the λ-core of M, which is defined like a core in the context of the ordinary core model K. We can reprove (37) and (38). The mice M^X, M^Y can be well-ordered via fine-structure preserving iterations like the core mice of K; we carry over (39) and (40) to the present situation. With this we can imitate the rest of the argument. Notice that in establishing the analogue of (46) one uses that M^X is a $\lambda^X_{\omega_1}$-core.

So, finally, we get a contradiction, and the assumption that no inner model contains ω_1 measurable cardinals is false. QED

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