

# An invitation to formalising mathematics

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Slides at [www2.mathematik.hu-berlin.de/~rothganm/](http://www2.mathematik.hu-berlin.de/~rothganm/)

# Outline of today's talk

- 1 What is a proof?
- 2 What is formalisation?
- 3 What has been formalised?
- 4 How to formalise?
- 5 Learning Lean

# What is a proof?

## Proof: formal definition

A mathematical proof is a sequence of *formal* logical deductions, starting from a set of axioms.

## Proof: practical definition

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## Proof: practical definition

A mathematical proof is a sequence of arguments convincing an educated reader. *In principle*, all details can be filled in.

Proof correctness is a social convention!

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## Example (Poincaré's, stability of the solar system)

Every single issue of Acta Mathematica retracted and reprinted.

## Example (Four-colour theorem)

Proofs by Kempe and Tait (around 1880) each believed correct — for 11 years.

## Example (Classification of finite simple groups)

Gap (quasi-thin case), only closed after 21 years



# Some papers are wrong

## Example (Baker's theorem, 1970)

- key lemma is false (Rempe-Sixsmith 2019)
- many papers using it can be fixed; another bunch is now open
- five much-cited papers “generalised” the argument

# Some papers are wrong

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- many papers using it can be fixed; another bunch is now open
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## Example (Hilbert's 21st problem)

“Proof” by Plemelj (1908) found wrong in 1970s  
solved in 1990 with different answer

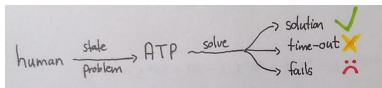
## Example (Hilbert's 16th problem, part 2)

Solution by Dulac (1923), found wrong in 1981



# What does formalisation mean? (cont.)

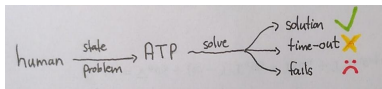
answer 2: automated theorem proving



problems: hit or miss; opaque

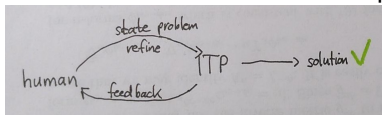
# What does formalisation mean? (cont.)

answer 2: automated theorem proving



problems: hit or miss; opaque

answer 3: interactive theorem proving



# Why formalise?



Verification



Creation



Understanding



Collaboration

# Why formalise?

- verification: peer reviewer's dream  
only check definitions and theorems make sense
- understanding: reader chooses amount of detail  
Demo by Patrick Massot and Kyle Miller:  
`https://www.imo.universite-paris-saclay.fr/~patrick.massot/Examples/ContinuousFrom.html`

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Demo by Patrick Massot and Kyle Miller:  
<https://www.imo.universite-paris-saclay.fr/~patrick.massot/Examples/ContinuousFrom.html>
- database of theorems: searching known and related results  
only requires *statements* of main results
- creation: can this lemma be generalised? unused assumptions?
- collaboration: less trust required



# What has been formalised already: let's guess

- Banach–Schauder open mapping theorem
- Birkhoff Ergodic Theorem
- Mandelbrot set is connected
- Cauchy-Kovalevskaya Theorem on existence of an analytical solution of an analytical PDE.
- Denjoy's theorem: a  $C^2$  orientation-preserving diffeomorphism of the circle with an irrational rotation number is conjugate to a rotation.
- Sphere eversion
- Existence of Haar measure
- Existence of a smooth partition of unity
- Feit–Thompson theorem/odd order theorem
- Fermat's Last Theorem
- Four colour theorem
- Galois correspondence
- Herman-Yoccoz theorem on linearization of a circle diffeomorphism
- Jordan curve theorem
- Liouville theorem: an entire holomorphic function is a constant
- Hilbert's Nullstellensatz
- Picard-Lindelöf theorem (existence and uniqueness of solutions of ODEs)
- Poincaré-Bendixson Theorem
- Poincaré recurrence theorem
- Sard's Theorem
- The continuum hypothesis is independent of ZFC.

## Let's guess: the answer

Only 5 are not formalised yet (AFAIK)

- Cauchy-Kovalevskaya Theorem on existence of an analytic solution of an analytic PDE
- Denjoy's theorem on rotation number
- Herman-Yoccoz theorem on linearization of a circle diffeomorphism
- Fermat's Last Theorem
- Sard's Theorem

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- Fermat's Last Theorem (in progress)
- Sard's Theorem (in progress)

# Notable formalisation projects

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fundamental lemma about condensed mathematics

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**took 3 weeks; complete before paper submitted**

## Some ongoing projects

- Almost Periodicity in Arithmetic Progressions
- Existence of an aperiodic monotile
- Prime Number Theorem (Kontorovich-Tao et al)
- Fermat's Last Theorem (Buzzard)

# A zoo of interactive theorem provers

- four widely used interactive theorem provers: Coq, Isabelle/HOL, Mizar and Lean
- large mathematics libraries: *mathcomp*, *Archive of formal proofs*, *Mizar Mathematical Library*, *mathlib*
- Coq: standard tool for software verification
- Isabelle: simple foundations, powerful automation
- Mizar: huge library
- Lean: newest (<10 years old), fast-growing

# Formalising research mathematics

- need a large library of mathematics
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- Why mathlib?
  - large integrated library
  - growing *fast*
  - system and tools are improving quickly
  - friendly and diverse community (github, zulip)

# Short demo

What is formalisation like?

- fussy; has learning curve
- it's fun — like a video game or programming
- makes you understand mathematics better

# Demo: backup in case of technical issues

Compact.lean ×

▽ □ …

Lean Infview ×

Mathlib &gt; Topology &gt; UniformSpace &gt; Compact.lean

```

155   intro y nxy
156   simp [comap_const_of_not_mem (compl_singleton_mem_nhds hxy) (Classical.not_not.2 rfl)]
157   #align uniform_space_of_compact_t2 uniformSpaceOfCompactT2
158
159   /-!
160   ### Heine-Cantor theorem
161   -/
162
163
164   /- Heine-Cantor: a continuous function on a compact uniform space is uniformly
165   continuous. -/
166   theorem CompactSpace.uniformContinuous_of_continuous [CompactSpace  $\alpha$ ] { $f$  :  $\alpha \rightarrow \beta$ }
167     (h : Continuous f) : UniformContinuous f :=
168   |calc map (Prod.map f f) ( $\mathcal{U} \alpha$ )
169     = map (Prod.map f f) ( $\mathcal{N}^s$  (diagonal  $\alpha$ )) := by rw [nhdsSet_diagonal_eq_uniformity]
170     _  $\leq \mathcal{N}^s$  (diagonal  $\beta$ )                    := (h.prod_map h).tendsto_nhdsSet mapsTo_prod_map_diagonal
171     _  $\leq \mathcal{U} \beta$                                 := nhdsSet_diagonal_le_uniformity
172   #align compact_space.uniform_continuous_of_continuous CompactSpace.uniformContinuous_of_continuous
173
174   /- Heine-Cantor: a continuous function on a compact set of a uniform space is uniformly
175   continuous. -/
176   theorem IsCompact.uniformContinuousOn_of_continuous { $s$  : Set  $\alpha$ } { $f$  :  $\alpha \rightarrow \beta$ } (hs : IsCompact s)
177     (hf : ContinuousOn f s) : UniformContinuousOn f s := by
178     rw [uniformContinuousOn_iff_restrict]
179     rw [isCompact_iff_compactSpace] at hs
180     rw [continuousOn_iff_continuous_restrict] at hf
181     exact CompactSpace.uniformContinuous_of_continuous hf
182   #align is_compact.uniform_continuous_on_of_continuous IsCompact.uniformContinuousOn_of_continuous
183

```

▼ Compact.lean:168:0

▼ Expected type

 $\alpha$  : Type  $u_1$  $\beta$  : Type  $u_2$  $\gamma$  : Type  $u_3$  $inst\ \dagger^2$  : UniformSp $inst\ \dagger$  : UniformSp $inst\ \dagger$  : CompactSpa $f$  :  $\alpha \rightarrow \beta$  $h$  : Continuous f

└ UniformContinuo

▶ All Messages (0)

# Learning Lean



# Learning Lean

- play the natural number game:
- textbook: mathematics in Lean  
`https://leanprover-community.github.io/mathematics\_in\_lean/index.html`
- further resources:  
`https://leanprover-community.github.io/learn.html`
- questions? ask on zulip

# Lean tutorials

- upcoming: Edinburgh, May 27-31 (women and mathematicians of minority gender); registration by Feb 26
- past/registration closed:
  - Düsseldorf (September 2023)
  - Regensburg (September 2023)
  - Rome (Jan 2024)
  - Marseille (March 2024)
  - Bonn (May 2024)
- up-to-date list: <https://leanprover-community.github.io/events.html>

# Lean in Berlin

- Lean study group, summer 2024 (email me if interested)
- Sebastian Pokutta, Tibor Szabó: Lean-related project
- Marc Kegel had a student using Lean
- ask your master's thesis advisor :-)

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Thanks for listening! Any questions?

# Comparing mathematical libraries: a closer look

- Archive of formal proofs: 4.1 million lines not integrated, articles are re-developing theory about half is “computer science” (e.g., properties of algorithms and programs)
- Coq’s library: different focus from standard mathematics (e.g., care about constructivism)
- MML: large and integrated; no statistics on size
- mathlib: 1.5 million lines, integrated

# My contributions to mathlib

Sard's theorem: prerequisites and reduction to normed spaces

- measure zero subsets of a manifold
- locally Lipschitz maps
- nowhere dense, meagre and sigma-compact sets
- local diffeomorphisms

Other mathematics

- interior and boundary of a manifold
- inverse function theorem for manifolds
- immersions, submersions and embeddings

Sphere eversion project: cleaning up, moving code into mathlib

Long-term vision: formalising the foundations of symplectic geometry